Final Exam, Math 205, Fall 2014

1. (i) Let \( \phi \in C^k(S^1) \) and \( \psi \in C^m(S^1) \). Show that the convolution \( \phi \ast \psi \) is in \( C^{k+m}(S^1) \). The convolution on the circle is defined as

\[
\phi \ast \psi(x) = \int_{S^1} \phi(x-y)\psi(y)dy.
\]

(ii) Give an example of a function \( \psi \in C(S^1) \) such that \( \psi \ast \psi \ast \cdots \ast \psi \) (\( k \) times) is not differentiable for any \( k \).

2. The Hausdorff-Young inequality claims that for \( 1 \leq p \leq 2 \) and \( 1/p + 1/q = 1 \) we have

\[
\| \hat{f} \|_{L_q(\mathbb{R})} \leq C \| f \|_{L_p(\mathbb{R})}.
\]

Here \( \hat{f} \) is the Fourier transform of \( f \). Show the converse: that is, if (1) holds for all \( f \in L^p(\mathbb{R}) \), then \( 1/p + 1/q = 1 \) and \( 1 \leq p \leq 2 \).

3. (a) Let \( B \subset C^1([0,1]) \) be a subspace of dimension \( N + 1 \). Show that there exists \( f \in B \) such that \( \sup_{0 \leq x \leq 1} |f(x)| = 1 \) and \( \sup_{0 \leq x \leq 1} |f'(x)| \geq 2N \). Hint: there has to be a function \( g \in B \) which vanishes sufficiently often.

(b) Prove that a subspace \( B \subset C^1([0,1]) \) which is closed under uniform convergence is finite dimensional. Hint: Show that \( \| f' \|_\infty \leq K\| f \|_\infty \) for some constant \( K \) and all \( f \in B \).

4. Let \( S \) be a subspace of \( C[0,1] \). Suppose \( S \) is closed as a subspace of \( L^2[0,1] \). Prove: (a) \( S \) is a closed subspace of \( C[0,1] \). (b) For \( f \in S \), \( \| f \|_2 \leq \| f \|_\infty \leq M\| f \|_2 \). (c) For every \( y \in [0,1] \), there is a \( K_y \in L^2[0,1] \) such that

\[
f(y) = \int K_y(x)f(x)dx
\]

for every \( f \in S \).

5. Let \( f \in L^1 \cap L^2(\mathbb{R}^n) \) and assume that \( \nabla f \in L^2(\mathbb{R}^n) \) (you may assume that \( f \in C^1(\mathbb{R}^n) \) and even that \( f \) is compactly supported if that would make it easier). Show that there exists a constant \( C_n > 0 \) that depends only on the dimension \( n \) so that

\[
\int |\nabla f|^2dx \geq C_n \left( \int |f|^2dx \right)^{1+2/n} \left( \int |f|dx \right)^{-4/n}.
\]

Hint: write \( \int |f|^2dx = \int |\hat{f}(\xi)|^2d\xi \), split the last integral into integrals over \( |\xi| \leq \rho \) and \( |\xi| \geq \rho \), evaluate each part separately and optimize over \( \rho \).

6. Let \( D \subset \mathbb{R}^2 \) be the strip \( x \in \mathbb{R}, y \in [0,1] \). Assume that \( f(x,y) \in C^1(D) \) tends to one uniformly as \( x \to -\infty \) and to zero as \( x \to +\infty \), and that \( 0 \leq f(x,y) \leq 1 \) for all \( (x,y) \in D \). Show that there exists a constant \( C > 0 \) so that for all such \( f \) we have

\[
\left( \int_D |\nabla f(x,y)|^2dxdy \right) \left( \int_D f(x,y)(1-f(x,y))dxdy \right) \geq C.
\]