Problem 1: Show that there exist two countable sub-collections $F_1, F_2$ of pairwise disjoint intervals, such that $F_1 \cup F_2$ covers $A$.

We'll first cover $A \cap (0, 1)$, then extend to the whole $\mathbb{R}$, so assume for now that $A \subset (0, 1)$. The strategy is to initially cover $A$ inductively by a countable collection of intervals that are not necessarily disjoint; afterwards we'll rearrange these intervals into 2 sub-collections, each of them disjoint.

Step 1. Constructing a countable cover for $A$.

Let $A_1 = A$, $G_1 = \{ I \in F : I \subset (0, 1) \text{ and center of } I \text{ is in } A_1 \}$, $\alpha_1 = \sup \{|I| : I \in G_1\} \leq 1$. If $\alpha_1 = 0$, there's nothing to prove. Otherwise, choose $I_1 \in G_1$ centered at $x_1 \in A_1$ with $|I_1| > 3/4\alpha_1$.

Given $A_n$, $G_n, I_n$ for $i = 1, \ldots, n - 1$, define $A_n = A \setminus \bigcup_{i=1}^{n-1} I_i$, $G_n = \{ I \in F : I \subset (0, 1) \text{ and center of } I \text{ is in } A_n \}$, and $\alpha_n = \sup \{|I| : I \in G_n\}$. If $\alpha_n = 0$, then $A \subset \bigcup_{i=1}^{n-1} I_i$ (remember, the intervals are non-degenerate). Otherwise again pick $I_n \in G_n$ centered at $x_n \in A_n$ with $|I_n| > 3/4\alpha_n$.

First, $\alpha_n \to 0$: In deed, if $\alpha_n = 0$ for some $n$, we're done. Otherwise $\alpha_{n+1} \leq \alpha_n$, so $\alpha_n \downarrow \alpha \geq 0$. If $m > n$, then $x_m \notin I_n$, so $|x_m - x_n| \geq |I_n|/2 \geq 3/8\alpha_n \geq 3/8\alpha$. Therefore we have an infinite sequence $x_n$ of elements in $(0,1)$ with distance between any two $\geq 3/8\alpha$, which can only happen if $\alpha = 0$.

Now we claim that $A \subset \bigcup I_n$. If not, let $x \in A \setminus \bigcup I_n$, and $I \subset (0,1)$ any interval in $F$ centered at $x$. Since $x \in A \setminus \bigcup I_n$, then for all $n$, $x \in A_n$, so $I \in G_n$, therefore $|I| \leq \alpha_n$. But $\alpha_n \to 0$, hence $|I| = 0$, contradicting the non-degeneracy assumption.

Step 2. Getting rid of the 'redundant intervals'.

We'll now get a new sub-collection $I'_n$ that has 'less' overlaps than the original one as follows: If $A \subset \bigcup_{i=1}^{\infty} I_n$, let $I'_1 = \emptyset$, otherwise $I'_1 = I_1$. In step $n$, if $A \subset \bigcup_{i=1}^{n-1} I'_i \cup (\bigcup_{i=n}^{\infty} I_i)$, let $I'_n = \emptyset$; otherwise $I'_n = I_n$. Then $A \subset \bigcup I'_n$, because by construction every point in $A$ is contained only in finitely many of the $I'_n$'s (if $x \in I_k$, then $\text{dist}(x, x_l) > 0$ for $l > k$, and $|I_n| \downarrow 0$), so we could not have removed all of them.

What we achieved this way is that at most two of the non-empty $I'_n$'s overlap at any point, because if $I_i, I_j, I_k$ all intersect, then one of them is included in the others, say $I_k$. But then $I'_1 = \emptyset$, contradicting the non-emptiness.

Step 3. Obtaining $F_1$ and $F_2$.

There are many ways to do this, but one nice way is using graph theory: let each $I'_n$ be a vertex of a (possibly infinite) graph, and connect two vertexes iff the corresponding intervals overlap. By the remark above, this can have no cycles, so it’s a tree, and hence bipartite. This means that the vertexes can be arranged into two sets $S_{1,2}$, each of them with no edges in between. Then put the intervals belonging to the set $S_1$ into $F_1$.

Step 4. Covering $A$ (not only $A \cap (0,1)$).

For each $n \in \mathbb{Z}$, pick an interval $J_n$ of radius $<1/2$ if $n \in A$, otherwise do nothing. $\mathbb{R} \setminus (\bigcup J_n)$ is a disjoint union of open intervals, each $\subset (n, n+1)$ for some $n$, so pick the disjoint collections $F^n_1, F^n_2$, also disjoint from the $J_n$'s. Then $F_1 = \bigcup_n F^n_1 \cup \{J_n\}_n$ and $F_2 = \bigcup_n F^n_2$.

Problem 2: Given an open set $U$, prove that there exists a countable disjoint collection of intervals $\mathcal{J}$ such that $\bigcup_{I \in \mathcal{J}} I \subset U$ and $\mu(A \setminus \bigcup_{I \in \mathcal{J}} I) = 0$.

Using problem 1, find collections $F_1, F_2$ (each disjoint) of intervals in $U_1 = U$ that cover $A \cap U$. Then $\mu(A \cap (\bigcup_{I \in \mathcal{J}} I) \geq \mu(A \cap U)/2$, say for $F_1$. Then there exist finitely many disjoint intervals $I_1, \ldots, I_{n_1}$ with $\mu(A \cap (\bigcup_{i=1}^{n_1} I_i)) \geq \mu(A \cap U) / 3$. Let $U_2 = U_1 \setminus (\bigcup_{i=1}^{n_1} I_i)$. This is again open, and $\mu(A \cap U_2) \leq 2/3\mu(A \cap U)$. Repeat the same with $U_2$ instead of $U_1$ to get $I_{n_1+1}, \ldots, I_{n_2}$ all disjoint from the previous $I_i$'s and each other, and with $\mu(A \cap (U_2 \setminus (\bigcup_{i=n_1+1}^{n_2} I_i))) \leq 2/3\mu(A \cap U_2) \leq (2/3)^2 \mu(A \cap U)$. Continuing this way, we obtain a disjoint sequence $I_1, I_2, \ldots$ that satisfies $\mu(A \setminus (\bigcup I_i)) \leq (2/3)^n \mu(A \cap U)$ for all $n$, therefore giving $\mu(A \setminus (\bigcup I_i)) = 0$.

Problem 3: Construct a monotone function that is discontinuous on a dense set on $[0, 1]$. 
Let \( q_1, q_2, ... \) be an enumeration of rationals, and let \( f(x) := \sum 2^{-n} \chi_{(q_n, \infty)} \). \( f \) is monotone, as a sum of increasing functions. Furthermore, it’s discontinuous in \( \mathbb{Q} \): given \( n \), for all \( x > q_n \), \( f(x) \geq 2^n + f(q_n) \).

\[ \square \]

**Problem 4:** Show that almost every \( x \) lies in at most finitely many of the \( E_k \)'s.

\[
\mu(\{ x \text{ lies in infinitely many } E_k \}) = \mu(E^c = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k) = \mu(E^c) = \lim_{n \to \infty} \mu(\bigcup_{k \geq n} E_k) \leq \lim_{n \to \infty} \sum_{k=n}^{\infty} \mu(E_k) = 0,
\]
since \( \sum \mu(E_k) < \infty \). Hence, almost every \( x \) lies in at most finitely many \( E_k \)'s.

\[ \square \]

**Problem 5:** (i) Show that \( \phi_t(g) \to g(0) \).
(ii) How much can we weaken the regularity assumptions on \( \phi \) and \( g \)?

(i) Let \( M = \sup |g(x)| < \infty \) (since \( g \) is \( C^\infty \) and with compact support), and let \( \epsilon > 0 \). Then there exists a \( \delta > 0 \) s.t. \( |x| < \delta \) implies \( |g(x) - g(0)| < \epsilon \). Now noting that by a change of variables \( \int \phi_t(x) dx = 1 \), we get:

\[
|\phi_t(g) - g(0)| = \left| \int \phi_t(x)(g(x) - g(0)) dx \right| \leq \int |\phi_t(x)| |g(x) - g(0)| dx + \int \phi_t(x)|g(x) - g(0)| dx
\]

\[
\leq \epsilon \int_{|x| < \delta} \phi_t(x) dx + 2M \int_{|x| \geq \delta} \phi_t(x) dx \leq \epsilon \int \phi_t(x) dx + 2M \int_{|y| \geq \delta/t} \phi(y) dy
\]

But \( \int \phi = 1 \) implies that \( \int_{|y| \geq \delta/t} \phi \to 0 \) as \( t \to 0 \). Hence, \( \lim_{t \to 0} |\phi_t(g) - g(0)| \leq \epsilon \). Since this is true for every \( \epsilon \), the result follows.

(ii) In the proof above, all we needed was for \( g \) to be bounded and continuous, and \( \phi \) continuous. (The integrals all make sense if \( g \) and \( \phi \) are continuous, and \( \int \phi_t = 1 \).)

\[ \square \]