1. (a) We claim that the smallest positive integer that does not belong to \( F \) is \( \beta^t + 1 \). We can see that all smaller integers \( k \) are in \( F \) by considering the expression

\[
\frac{m}{\beta^t \beta^e}
\]

with \( e = t \) and \( m = k \).

Now suppose to the contrary that \( \beta^t + 1 \in F \). Then for some positive integer \( m \leq \beta^t \) and some integer \( e \) we have

\[
\beta^t + 1 = \frac{m}{\beta^t \beta^e}
\]

Rewrite this as

\[
1 = m\beta^{e-t} - \beta^t
\]

Since \( m \leq \beta^t \), we must have \( e > t \) (if \( e \leq t \), then \( \beta^{e-t} \leq 1 \), so \( m\beta^{e-t} - \beta^t \leq m - \beta^t \leq 0 \), contradiction). But then we can write

\[
1 = \beta(m\beta^{e-t-1} - \beta^{t-1})
\]

where the term in parentheses is an integer since \( t \geq 1 \). But \( \beta \) is an integer greater than or equal to 2, so this is a contradiction.

(b) For IEEE single and double precision arithmetic, we get \( 2^{24} + 1 \) and \( 2^{53} + 1 \) respectively (quite large numbers).

2. Let’s first apply the classical Gram–Schmidt algorithm. We start with \( v_1 = a_1, r_{11} = \sqrt{1 + \epsilon^2} \approx 1 \), so

\[
q_1 = \frac{a_1}{r_{11}} = \begin{pmatrix} 1 \\ \epsilon \\ 0 \\ 0 \end{pmatrix}.
\]

Next \( r_{12} = a_1^*a_2 = 1 \). Then

\[
v_2 = a_2 - r_{12}q_1 = \begin{pmatrix} 0 \\ -\epsilon \\ \epsilon \\ 0 \end{pmatrix}.
\]

Then \( r_{22} = \sqrt{\epsilon^2 + \epsilon^2} = \epsilon \sqrt{2} \), so \( q_2 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \). Next \( r_{13} = q_1^*a_3 = 1 \), while \( r_{23} = q_2^*a_3 = 0 \), so

\[
v_3 = a_3 - 1q_1 - 0q_2 = \begin{pmatrix} 0 \\ -\epsilon \\ 0 \\ \epsilon \end{pmatrix}.
\]
and hence \( r_{33} = \varepsilon \sqrt{2} \) and 
\[
q_3 = \begin{pmatrix}
0 \\
-1/\sqrt{2} \\
0 \\
1/\sqrt{2}
\end{pmatrix}.
\]

Now notice that \( q^*_2 q_3 = 1 \), though in theory this inner product should be zero!

Now we apply the modified Gram–Schmidt algorithm. We can start by setting every \( v^{(1)}_j = a_j \). As before \( v_1 = v^{(1)}_1 = a_1 \) and \( r_{11} \approx 1 \), so 
\[
q_1 = a_1 / r_{11} = \begin{pmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{pmatrix}.
\]

Next \( v^{(2)}_2 = v_2 - (q^*_1 v^{(1)}_2) q_1 = a_2 - (q^*_1 a_2) q_1 \). We get \( r_{12} = q^*_1 a_2 = 1 \) exactly, so 
\[
v_2 = v^{(2)}_2 = \begin{pmatrix} 0 \\ -\varepsilon \\ \varepsilon \\ 0 \end{pmatrix}
\]
and
\[
r_{22} = \varepsilon \sqrt{2}, \quad q_2 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}
\]
as in the classical algorithm. We get \( v^{(3)}_3 = v^{(1)}_3 - (q^*_1 v^{(1)}_3) q_1 = a_3 - (q^*_1 a_3) q_1 \). Here \( r_{13} = q^*_1 a_3 = 1 \) exactly, so 
\[
v^{(2)}_3 = v^{(3)}_3 = \begin{pmatrix} 0 \\ \varepsilon \\ 0 \\ -\varepsilon \end{pmatrix}.
\]
Finally \( v^{(3)}_3 = v^{(2)}_3 - (q^*_2 v^{(2)}_3) q_2 \). We get \( r_{23} = q^*_2 v^{(2)}_3 = \varepsilon / \sqrt{2} \) so we compute 
\[
v_3 = v^{(3)}_3 = \begin{pmatrix} 0 \\ -\varepsilon/2 \\ -\varepsilon/2 \\ \varepsilon \end{pmatrix}
\]
and finally
\[
r_{33} = \varepsilon \sqrt{\frac{3}{2}}, \quad q_3 = \begin{pmatrix} 0 \\ -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}.
\]

We can check that \( q^*_1 q_2, q^*_1 q_3, \) and \( q^*_2 q_3 \) are all zero, showing that these vectors form an orthogonal set, unlike the \( q_i \)’s gotten via the classical Gram–Schmidt algorithm.

3. (a) The characteristic polynomial of \( M = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \) is \( p_M(\lambda) = \lambda^2 - 2a \lambda + (a^2 + b^2) \) so we find that the eigenvalues are \( a \pm bi \). We can find that an eigenvector for \( a + bi \) is \( (1, i)^T \), while
an eigenvector for $a - bi$ is $(1, -i)^T$. Thus we can write $M = X\Lambda X^{-1}$ for $\Lambda = \begin{pmatrix} a + bi & 0 \\ 0 & a - bi \end{pmatrix}$ and $X = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. From the formula for the inverse of a $2 \times 2$ matrix, we get $X^{-1} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$.

Hence

$$M = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} a + bi & 0 \\ 0 & a - bi \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

is a diagonalization of $M$. If some $2 \times 2$ $A$ has eigenvalues $a \pm bi$, then since the eigenvalues are distinct, $A$ is diagonalizable. (This last implication holds in general, but it is easier to prove in the case of just a $2 \times 2$ matrix—try showing this in the $2 \times 2$ case, if you didn’t already think about it while doing this problem.) Thus $A = Y\Lambda Y^{-1}$ for some $Y$. We have $M = X\Lambda X^{-1}$ which is $\Lambda = X^{-1}MX$. Hence $A = YX^{-1}MXY^{-1} = (YX^{-1})M(YX^{-1})^{-1}$, showing that $A$ is similar to $M$.

(b) We see that the two columns of $M$ are orthogonal to each other, so up to scaling by some factor, $M$ can be described as a rigid motion. Thus it is a composition of the rotation which sends $e_1$ to a vector in the direction $(a, -b)^T$ and scaling by a factor of $\sqrt{a^2 + b^2}$ (the order doesn’t matter since a linear transformation which just scales by some factor commutes with any other linear transformation).

4. (a) Algebraically, for any Householder reflector $Q$, we have $Q^2 = I$, so if $Qv = \lambda v$, then $v = Q^2v = Q(\lambda v) = \lambda(Qv) = \lambda^2v$. So $v = \lambda^2v$ and hence $\lambda^2 - 1 = 0$, so $\lambda = \pm 1$, i.e., all the eigenvalues of $Q$ are either 1 or $-1$.

Geometrically, let $H$ be the $(m - 1)$-dimensional subspace which $Q$ reflects across. Any vector in $H$ is sent by $Q$ to itself, so these are all eigenvectors for the eigenvalue 1. A vector $v$ perpendicular to $H$ is sent to $-v$ so such a vector is an eigenvector for the eigenvalue $-1$. By choosing a basis for $H$ and a vector perpendicular to $H$ we can construct a basis of eigenvectors corresponding to these two eigenvalues, so $\pm$ are all the eigenvalues.

(b) From the geometric interpretation of part (a), we know that the eigenvalue +1 appears with algebraic multiplicity at least $m - 1$ since the eigenspace is $(m - 1)$-dimensional. (We’re using that “algebraic multiplicity” is at least “geometric multiplicity; see lecture 24 in the book for details.) Since the eigenvalue $-1$ has multiplicity $\geq 1$, we conclude $p_Q(\lambda) = (\lambda - 1)^{m-1}(\lambda + 1)$. Since det $Q$ is the product of the eigenvalues (with multiplicity), det $Q = -1$.

(c) Since $Q$ is hermitian, its singular values are just the absolute values of its eigenvalues. So all $m$ singular values of $Q$ are 1. Even more directly, $Q$ is unitary, so $Q = QII$ is an SVD, which shows all the singular values are 1.

5. Solution to bonus problem to appear later...