Note: we only briefly mentioned in class how to compute the SVD. The textbook lists the algorithm on p. 234. You may find it useful in problems 1 and 3.

1. (Trefethen–Bau 4.1 a,c,d,e) Determine the SVD’s of the following matrices (by hand calculation):
   
   (a) \[
   \begin{pmatrix}
   3 & 0 \\
   0 & -2
   \end{pmatrix}
   \]
   
   (c) \[
   \begin{pmatrix}
   0 & 2 \\
   0 & 0
   \end{pmatrix}
   \]
   
   (d) \[
   \begin{pmatrix}
   1 & 1 \\
   0 & 0
   \end{pmatrix}
   \]
   
   (e) \[
   \begin{pmatrix}
   1 & 1 \\
   1 & 1
   \end{pmatrix}
   \]

2. (Trefethen–Bau 4.4) Call two matrices \( A, B \in \mathbb{C}^{m \times m} \) unitarily equivalent if \( A = QBQ^* \) for some unitary \( Q \). Is it true or false that \( A \) and \( B \) are unitarily equivalent if and only if they have the same singular values? Explain your answer. (Be sure to check both directions.)

3. (Trefethen–Bau 5.3a–f) Let \( A = \begin{pmatrix} -2 & 11 \\ -10 & 5 \end{pmatrix} \)

   (a) Though we did not prove this, it is true that if a matrix has real entries, then it has an SVD where all the matrices have real entries. Determine (on paper) a real SVD of \( A \) in the form \( A = U \Sigma V^T \). Such an SVD is not unique, so find the one with the minimal number of minus signs in \( U \) and \( V \).

   (b) List the singular values, left singular vectors, and right singular vectors of \( A \). Draw a picture of the unit ball in \( \mathbb{R}^2 \) and its image under \( A \) together with the singular vectors.

   (c) What are the 1-, 2-, \( \infty \)- and Frobenius norms of \( A \)?

   (d) Find an expression for \( A^{-1} \) in terms of \( U, V, \) and \( \Sigma \), and use this to find \( A^{-1} \).

   (e) Find the eigenvalues \( \lambda_1, \lambda_2 \) of \( A \).

   (f) Verify that \( \det A = \lambda_1 \lambda_2 \) and that \( |\det A| = \sigma_1 \sigma_2 \).

4. Suppose \( A \in \mathbb{C}^{m \times m} \) has an SVD \( A = U \Sigma V^* \). Diagonalize the \( 2m \times 2m \) hermitian matrix

   \[ B = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \]

   where each entry above denotes an \( m \times m \) matrix. (Hint: first consider what \( B \) does to vectors of the form \( \begin{pmatrix} x \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ x \end{pmatrix} \) where \( x = u_i \) or \( v_i \), and then construct eigenvectors of \( B \) out of such vectors.)

5. Let

   \[ A = \begin{pmatrix} 1 & 1 \\ 0 & 10^{-10} \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \]

   and let

   \[ \delta A = \begin{pmatrix} 0 & 0 \\ 10^{-10} & -10^{-10} \end{pmatrix} \]

   (a) Solve \( Ax = b \). Solve \( (A + \delta A)y = b \). Compare the solutions \( x \) and \( y \).

   (b) Find the 1-, \( \infty \)-, and Frobenius norms of \( A \). You may give approximate answers.
(c) Find $A^{-1}$ (using any method you like) and find the 1-, $\infty$-, and Frobenius norms of $A^{-1}$ and hence the condition numbers $\kappa(A)$ with respect to these norms. Again, approximate answers are enough. Comment briefly on how this relates to your answer in (a).

6. For any invertible $m \times m$ matrix $A$, show that $\|A\|_2\|A^{-1}\|_2 \geq 1$. (Hint: you may use the fact that since the unit sphere in the 2-norm is closed and bounded, there exists a vector $u$ with $\|u\|_2 = 1$ such that $\|Au\|_2 = \|A\|_2$. You only need this fact and the definition of induced matrix norm. In fact, the result holds for any $p$-norm, not just for $p = 2$.)