Math 104, Summer 2010
Final Exam

Instructions: You may not use any books, notes, or calculators, or electronic devices. Write all your answers in the blue book. Be sure to make clear how you arrive at your answers. You have 3 hours; it may be best to first do the problems which you find the easiest.

1. (15 pts.) Suppose $A$ is a $2 \times 2$ matrix with two distinct eigenvalues $\lambda \neq \mu$. Show that $A$ is diagonalizable.

2. (25 pts. total, 5 pts. for each part)
   (a) Show that if $Q \in \mathbb{C}^{m \times m}$ is unitary, then $Q$ preserves inner products, i.e., show that $(Qx)^*(Qy) = x^*y$ for all $x, y \in \mathbb{C}^m$.
   (b) Deduce that a unitary $Q$ preserves lengths, i.e., if $\theta$ is the angle between $x$ and $y$ and $\phi$ is the angle between $Qx$ and $Qy$, then $\phi = \pm \theta$.
   (c) Show that if any matrix $Q \in \mathbb{C}^{m \times m}$ preserves lengths and angles, then $Q$ preserves inner products. In other words, show that if $\|Qx\| = \|x\|$ for all $x \in \mathbb{C}^m$ and (with notation as in (b)) $\phi = \pm \theta$ for all $x, y \in \mathbb{C}^m$, then $(Qx)^*(Qy) = x^*y$ for all $x, y \in \mathbb{C}^m$.
   (d) Show that if a matrix $Q \in \mathbb{C}^{m \times m}$ preserves the inner product, i.e., $(Qx)^*(Qy) = x^*y$ for all $x, y \in \mathbb{C}^m$, then $Q$ is unitary. (Hint: make a judicious choice of $x$ and $y$.)
   (e) If $Q$ is unitary, what are the singular values of $Q$? Briefly explain.

3. (20 pts.) Suppose $A, B \in \mathbb{C}^{m \times m}$. Determine which of the following inclusions always hold:
   (a) range$(AB) \supseteq$ range$(A)$
   (b) range$(AB) \subseteq$ range$(A)$
   (c) range$(AB) \supseteq$ range$(B)$
   (d) range$(AB) \subseteq$ range$(B)$
   In each case, if the statement is always true, prove it; if not, provide a counterexample. (Hint: try some simple matrices with $m \leq 2$.)

4. (20 pts. total, 5 pts. for each part)
   (a) Determine a singular value decomposition of the matrix
      \[ A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \]
   (b) Determine a singular value decomposition of the matrix
      \[ B = \begin{pmatrix} 2 & -2 \\ 2 & 4 \\ -1 & 4 \end{pmatrix} \]
   (Hint: what do you notice about the columns of $B$?)
   (c) Given $x = (x_1, x_2, x_3)^T$, find the maximum possible value of $\|Ax\|_2$ in terms of $x_1, x_2, x_3$.
   (d) Given $x = (x_1, x_2)^T$, find the maximum possible value of $\|Bx\|_2$ in terms of $x_1, x_2$.

5. (20 pts. total) Let $v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Let $H = \text{span}\{v\}^\perp$.
   (a) (5 pts.) Find the matrix for $P_v$ the orthogonal projection onto $\text{span}\{v\}$. 

(b) (5 pts.) Find the matrix for $P_H$, the orthogonal projection onto $H$.

(c) (10 pts.) Find $X$ and $\Lambda$ such that $P_H = X\Lambda X^{-1}$. (Hint: find a basis for $H$, and use that $P_H$ is an orthogonal projector.) You do not need to compute $X^{-1}$.

6. (20 pts. total, 10 pts. for each part)
   (a) Find the 1-norm, 2-norm, $\infty$-norm, and Frobenius norm of the matrix
   \[ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \]
   (b) Find the condition number $\kappa(A) = \|A\|\|A^{-1}\|$ in each of the four norms listed in part (a).

7. (20 pts. total) Suppose we have the following input data points in the $xy$-plane and output data values in the $z$ coordinate:
   \[
   (x_1, y_1) = (1, 0) \quad z_1 = 3 \\
   (x_2, y_2) = (0, 1) \quad z_2 = 2 \\
   (x_3, y_3) = (-1, 0) \quad z_3 = 5 \\
   (x_4, y_4) = (0, -1) \quad z_4 = 4
   \]
   Suppose we want to find a plane $z = f(x, y)$ in $\mathbb{R}^3$ which best approximates this data; that is, we write $f(x, y) = ax + by + c$ and want to find which values of $a, b, c$ will best fit the data.
   (a) (5 pts.) Write a matrix-vector equation $Ax = b$ corresponding to $f(x_i, y_i) = z_i$ for all $i = 1, ..., 4$.
   (b) (15 pts.) Using the method of least squares applied to the system of equations in part (a), find the plane that best fits these data points.

8. (BONUS, 15 pts.) Let $A$ be a $3 \times 3$ matrix. Describe how to construct Householder reflectors $Q_1, ..., Q_k$ such that multiplying by $A$ by these matrices results in a lower-triangular matrix
   \[
   R = \begin{pmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{pmatrix}
   \]
   Be sure to specify which order to multiply the matrices in. You do not need to calculate explicit formulae, though your outline should make clear how such formulae could be found. Also do not worry about minimizing errors due to rounding on a computer.