Problem 1. In a fictional soccer match-up between the national teams of Argentina (A), Belgium (B), and Croatia (C), the following results were recorded:

\[ \text{A vs. B: 2 - 0} \quad \text{A vs. C: 3 - 1} \quad \text{B vs. C: 0 - 1}. \]

We form a matrix \( M \) with the ratios “team’s score” over “total score”,

\[
M = \begin{pmatrix}
0 & \frac{3}{4} & 0 \\
\frac{1}{4} & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & 0
\end{pmatrix}.
\]

For instance, the element (1, 3) or (A, C) is equal to \( \frac{3}{4} \) because A scored 3 of the 4 goals in the A vs. C match. We arbitrarily chose to put zeros on the diagonal.

Consider the eigenvector \( \mathbf{v} \) corresponding to the largest (real-valued) eigenvalue of \( M \). The components \( v_A, v_B, v_C \) of this eigenvector are indicators of the performance of the respective teams: the higher the number, the better the team. So these numbers provide a pretty good, automatic way of ranking the teams.

(a) (5 pts.) Compute the eigenvalues of \( M \).

The characteristic polynomial is

\[
\det \left( \begin{pmatrix} -\lambda & 3/4 & 0 \\
1/4 & -\lambda & 0 \\
0 & 0 & -\lambda \end{pmatrix} \right) = \lambda^3 - \frac{3}{4} \lambda^2 - \frac{3}{2} \lambda.
\]

The eigenvalues are \( 0 \), \( \frac{\sqrt{3}}{2} \), \( -\frac{\sqrt{3}}{2} \).
(b) (5 pts.) Compute the normalized eigenvector \( v \) of \( M \) corresponding to the largest eigenvalue of \( M \). Check that the values obtained for the components of \( v \) obey \( v_A > v_C > v_B \), hence reproduce the expected ranking "A is better than C, and C is better than B".

\[
\text{The largest eigenvalue is } \frac{\sqrt{3}}{4}.
\]

The eigenvector is
\[
\begin{pmatrix}
V_A \\
V_B \\
V_C
\end{pmatrix}
\]

\[
\begin{align*}
V_A &\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + V_B \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + V_C \begin{pmatrix} 3/4 \\ 0 \\ 0 \end{pmatrix} = \frac{\sqrt{3}}{4} \begin{pmatrix}
V_A \\
V_B \\
V_C
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
V_B + \frac{3}{4} V_C &= \frac{\sqrt{3}}{4} V_A \\
V_B &= 0 \\
\frac{1}{4} V_A + V_B &= \frac{\sqrt{5}}{4} V_C
\end{align*}
\]

\[
\sqrt{3} V_C = V_A
\]

\[
\frac{1}{\sqrt{14}} \begin{pmatrix}
\sqrt{3} \\
0 \\
1
\end{pmatrix} = \frac{\sqrt{3}}{4} \begin{pmatrix}
\sqrt{3}/2 \\
0 \\
\sqrt{2}
\end{pmatrix} = u^-
\]

\[
V_A > V_C > V_B \text{ is verified}
\]
(c) (5 pts.) Consider $M$ defined with some other number than zero on the diagonal, say

$$
M_\mu = \begin{pmatrix}
\mu & 1 & \frac{3}{4} \\
0 & \mu & 0 \\
\frac{1}{4} & 1 & \mu
\end{pmatrix}.
$$

Would this alternative choice have changed the ranking of the teams? In your answer, justify how $\mu$ affects the eigenvalues and eigenvectors of $M_\mu$ compared to those of $M$.

Let $\mathbf{v}$ be an eigenvector of $\Pi$

Then $\mathbf{v}$ is an eigenvector of $\Pi_\mu$

eigenvalue $\mu + \mu$

$\Pi_\mu \mathbf{v} = \mu \mathbf{v} + \mu \mathbf{v} = (\mu + \mu) \mathbf{v}.$
Problem 2. (10 pts.) Prove that the product of two \( n \)-by-\( n \) upper-triangular matrices is still upper-triangular, by induction over \( n \). [Hint: for the induction step, decompose each \( (n+1) \)-by-\( (n+1) \) matrix into four blocks of your choosing. For matrix-matrix multiplication, the rule of going along rows of the first matrix and down columns of the second matrix also works with blocks when their sizes properly match.]

\[
\begin{pmatrix}
A & & & a_1 & \\
& & & & \\
& & & & \\
& & & & \\
o & & & a_{n-1} & \\
\end{pmatrix}
\begin{pmatrix}
B & & & b_1 & \\
& & & & \\
& & & & \\
& & & & \\
o & & & b_{n-1} & \\
\end{pmatrix}
= \begin{pmatrix}
AB & & & ? & \\
& & & & \\
& & & & \\
& & & & \\
o & & & a_{n-1} & \\
\end{pmatrix}
\]

\( A \) upper triangular of size \( n-1 \)
\( B \)
so by induction \( AB \) upper triangular
so
\[
\begin{pmatrix}
AB & & & ? & \\
& & & & \\
& & & & \\
& & & & \\
o & & & a_{n-1} & \\
\end{pmatrix}
\] is upper triangular.
Problem 3.

(a) (6 pts.) Fill in with any choice of "range space", "row space", "nullspace", or "left-nullspace", such that the following sentences are true:

- The relation $A^2 = 0$ implies that the ____range space____ of $A$ is included in the nullspace of $A$.
- The nullspace is the orthogonal complement of the ____row space____.
- The dimension of the ____range space____ is equal to the dimension of the ____row space____, and is equal to the rank.

(b) (4 pts.) Prove that if $A$ is an $n$-by-$n$ matrix such that $A^2 = 0$, then necessarily $\text{rank}(A) \leq n/2$. [Hint: use all three assertions of part (a).]

\[
\begin{align*}
\text{range space } A & \subseteq \text{null } A \\
\dim \text{range space } A & \leq \dim \text{null } A \\
\dim \text{null } A + \dim \text{row space} & = n \\
\dim \text{range space } A + \dim \text{row space} & \leq n \\
\Rightarrow \dim \text{rank } A & \leq \frac{n}{2}
\end{align*}
\]
Problem 4. Let $S \in \mathbb{C}^{m \times m}$ be skew-hermitian, i.e., $S^* = -S$.

(a) (5 pts.) Show that the eigenvalues of $S$ are pure imaginary, i.e., are complex numbers with the real part equal to zero. [Hint: contrast with the situation of hermitian matrices $A^* = A$, as in problem 2 of Homework 4.]

Let $\lambda$ be an eigenvalue with eigenvector $\mathbf{u}$.

$$S \mathbf{u} = \lambda \mathbf{u}$$

$$\overline{S} \mathbf{u} = \overline{\lambda} \mathbf{u} = \mathbf{u}^* (S^* \mathbf{u}) = \mathbf{u}^* (-S \mathbf{u}) = -\lambda \mathbf{u} \implies \overline{\lambda} = -\lambda \text{ are pure imaginary}$$

(b) (5 pts.) Show that $I - S$ is invertible. [Hint: among the various criteria for invertibility, consider the one involving eigenvalues.]

$I$ is not an eigenvalue so $I - S$ is invertible.

(c) (5 pts.) Show that the matrix $Q = (I - S)^{-1}(I + S)$, known as the Cayley transform of $A$, is unitary.

$$Q^*Q = (I + S)^+ (I - S^+) = (I + S)(I - S^+)$$

$$Q^*Q = (I - S)^{-1} (I + S) (I - S^+) = (I - S)^{-1} (I + S)$$

$$= (I - S)^{-1} (I + S) (I + S)^{-1} = (I - S)^{-1} (I - S^2) (I + S)^{-1}$$

$$= (I - S)^{-1} (I - S) (I + S)^{-1} = I$$
Problem 5.

(a) (5 pts.) Prove that the eigenvalues of a projector $P$ can have no other value than zero or one.

\[ P^2 = P \quad \text{let } d \text{ be eigenvalue, } \nu \text{ eigenvecctor } \nu \neq 0 \]

\[ P^2 \nu = P \nu \]

\[ d^2 \nu = d \nu \]

\[ d^2 = 1 \implies d \in \{0, 1\} \]

(b) (5 pts.) Prove that the singular values of an orthogonal projector $P$ can have no other value than zero or one.

\[ P^* = P \quad \text{the singular value are the eigenvalue of} \]

\[ P^* P = P^2 = P \]

\[ \text{singularvalue } \in \{ \sqrt{0}, \sqrt{1} \} = \{0, 1\} \]
Problem 6. (10 pts.) Prove the inequality

\[ \|A\|_F \leq \sqrt{\text{rank}(A)} \|A\|_2. \]

[Hint: use the SVD.]

\[ A = U \Sigma V^* \quad \text{U, V orthogonal} \]

\[ \|A\|_F = \|U \Sigma V^*\|_F = \|\Sigma V^*\|_F \]

\[ \Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix} \quad r = \text{rank}(A) \]

\[ V^* = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \]

\[ \Sigma V^* = \begin{pmatrix} \sigma_1 v_1 \\ \sigma_2 v_2 \\ \vdots \\ \sigma_r v_r \end{pmatrix} \]

\[ \|\Sigma V^*\|_F^2 = \sum \sigma_i^2 (p_{i1}^2 + \cdots + p_{in}^2) + \cdots + \sigma_r^2 (p_{r1}^2 + \cdots + p_{rn}^2) \]

\[ \leq \sigma_1^2 \left( \|p_1\|_2^2 + \cdots + \|p_n\|_2^2 \right) + \cdots + \sigma_r^2 \left( \|p_1\|_2^2 + \cdots + \|p_n\|_2^2 \right) \]

\[ = \sigma_1^2 \sum \|p_i\|_2^2 = \|A\|_F \quad \text{since U is orthogonal} \]

\[ \|\Sigma V^*\|_F \leq \sigma_1 \]

\[ \|\Sigma V^*\|_F \leq \sigma_1 \sqrt{\text{rank}(A)} \leq \frac{\sqrt{\text{rank}(A)} \|A\|_2}{8} \quad \text{since} \quad \|A\|_F = \sigma_1 \]

\[ \|\Sigma V^*\|_F \leq \frac{\sqrt{\text{rank}(A)} \|A\|_2}{8} \]

\[ \|\Sigma V^*\|_F \leq \frac{\sqrt{\text{rank}(A)} \|A\|_2}{8} \quad \text{since} \quad \|A\|_F = \sigma_1 \]
Problem 7. Consider the matrix

\[ A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}. \]

(a) (8 pts.) Compute the reduced QR decomposition \( A = \hat{Q} \hat{R} \). [Hint: use Gram-Schmidt to obtain \( \hat{Q} \), and then use \( \hat{R} = \hat{Q}^* A \) to obtain \( \hat{R} \).]

\[ u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \]

\[ u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \]

Gram Schmidt

\[ \hat{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}, \]

\[ \hat{R} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}. \]

\[ \hat{R} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]

Given two vectors

\[ u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \]

\[ u_2 = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}. \]
(b) (7 pts.) Using your result in part (a), solve $Ax = b$ in the least squares sense, where

$$b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{R}^\top \hat{Q} \hat{R} \hat{x} = \hat{R}^\top \hat{Q} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(2) $\hat{R}^\top \hat{R} \hat{x} = \hat{R}^\top \hat{Q} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

To find one solution, just let us solve

$$\hat{R} \hat{x} = \hat{Q} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\frac{\sqrt{5}}{\sqrt{5}} x_2 = -\frac{1}{\sqrt{6}} \quad x_2 = -1$$

$$\sqrt{2} x_1 + \frac{1}{\sqrt{2}} x_2 = \frac{1}{\sqrt{2}}$$

$$x_1 = \frac{1}{\sqrt{2}}$$

$$x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$Ax = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
Problem 8.

(a) (5 pts.) Assume that $A$ is an invertible square matrix. Let $\|A\|$ be any induced norm of $A$. Prove that the condition number

$$\kappa(A) = \|A\| \|A^{-1}\|$$

is always greater than or equal to one. [Hint: don't specialize to a special choice of norm, just use properties of induced norms.]

$$\|AA^{-1}\| = 1$$

$$\|AA^{-1}\| \leq \|A\| \|A^{-1}\|$$

$$\infty \quad 1 \leq \|A\| \|A^{-1}\|$$
(b) (5 pts.) Compute $\kappa_\infty(A)$ when the norm is chosen as the infinity norm, and

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad A^{-1} = \frac{1}{2} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}.$$  

Verify that indeed $\kappa_\infty(A) \geq 1$.

$$\| A \|_\infty = \max \left( |1| + |2|, |3| + |4| \right)$$

$$= 5$$

$$\| A^{-1} \|_\infty = \max \left( \frac{1}{2} (|1| + |2|), \frac{1}{2} (|3| + |1|) \right)$$

$$= 3$$

$\kappa_\infty(A) = 15$
Problem 9. (5 pts.) Let $A, B \in \mathbb{R}^{m \times n}$, with $m \geq n$, such that range$(A) = $ range$(B)$. Show that $B^*A$ is invertible if and only if $A$ is full column-rank. [Hint: contrast with problem 5 of Homework 5.]

\[ B^*A \text{ invertible } \iff \text{null } (B^*A) = \{0\} \]

\[ \text{since } B^*A \text{ is a square matrix} \]

\[ \exists x \in \text{null } (B^*A) \iff Ax \in \text{null } B^* = (\text{range } B) \perp \\
\quad = (\text{range } A) \perp \\
\quad \text{Ax} \in \text{range } A \text{ and } (\text{range } A) \perp \\
\iff Ax = 0 \\
\iff x \in \text{null } A. \]