3.2.1. Since $M_g$ is connected, we know that $H^0(M_g) \cong \mathbb{Z}$ and that the generator 1 of this group is the identity in the cup product structure. Also, we know that $H_2(M_g) \cong \mathbb{Z}$, so $H^2(M_g) \cong \mathbb{Z}$ by the universal coefficients theorem for cohomology. Since $H^n(M_g) = 0$ for $n > 2$, we know that $\theta \in H^2(M_g)$, $\theta \sim 1 = \theta$, while its cup product with an element in any other dimension must be 0. So it remains to compute the cup product of elements in $H^1(M_g) \cong \mathbb{Z}^{2g}$.

The quotient map induces a map $H^*(\bigvee^g M_1) \to H^*(M_g)$. By naturality of the cup product, we have the following commutative diagram:

$$
\begin{array}{ccc}
H^1(\bigvee^g M_1) \times H^1(\bigvee^g M_1) & \xrightarrow{\cup} & H^2(\bigvee^g M_1) \\
\downarrow & & \downarrow \\
H^1(M_g) \times H^1(M_g) & \xrightarrow{\cup} & H^2(M_g)
\end{array}
$$

In the top row, we know that if $a^*_i, b^*_j$ are generators for $H^1$ and $c^*_i$ is a generator of $H^2$ of the $i$th copy of $M_1$, then $a^*_i \prec b^*_j = \delta_{ij} c^*_i$, while $a^*_i \prec a^*_j = b^*_i \prec b^*_j = 0$ for all $i, j$. That this is the cup product on the wedge sum follows from the ring isomorphism

$$
\tilde{H}^*(\bigvee X_\alpha) \cong \Pi_\alpha \tilde{H}^*(X_\alpha)
$$

from p.215 in Hatcher.

If $c_i$ represent generators of $H_2(\bigvee M_1)$ and $C$ a generator of $H_2(M_g)$, then the quotient map induces the map $C \rightarrow c_1 + \ldots + c_g$ at the chain level (this is seen geometrically). (Note - Cochains representing generators of $H^2$ are just the duals of these elements by the universal coefficients theorem, so our notation is justified.) The map induced on cochains (and hence on cohomology) is $c^*_i \mapsto C^*$ for all $i$. Similarly if $A_i, B_i$ represent generators of $H_1(M_g)$ and $a_i, b_i$ generators of $H_1(\bigvee M_1)$, the quotient map induces maps $A_i \mapsto a_i$ and $B_i \mapsto b_i$ on homology class representatives, so it induces $a^*_i \mapsto A^*_i$, $b^*_i \mapsto B^*_i$ on cochains and cohomology.

Thus $A^*_i \prec B^*_j = \delta_{ij} C^*$, while $A^*_i \prec A^*_j = B^*_i \prec B^*_j = 0$.

3.2.2. From the long exact sequences of the pairs $(X, A)$ and $(X, B)$, we get that $H^*(X; R) \cong H^*(X, A; R)$ and $H^*(X; R) \cong H^*(X, B; R)$ in every positive dimension (since every third term $H^*(A)$ or $H^*(B)$ is zero). By naturality of the cup product, we get the following commutative diagram:

$$
\begin{array}{ccc}
H^k(X, A; R) \times H^\ell(X, B; R) & \xrightarrow{\cup} & H^{k+\ell}(X, A \cup B; R) \\
\downarrow \cong & & \downarrow \\
H^k(X; R) \times H^\ell(X; R) & \xrightarrow{\cup} & H^{k+\ell}(X; R)
\end{array}
$$

Since the upper right-hand term is 0, the commutativity implies that the cup product in the bottom must be 0. This clearly applies to a suspension of a space, since that is the union of two (neighborhoods of) cones on the space, which are contractible.

In the case that $X = A_1 \cup \ldots \cup A_n$ with each $A_i$ a contractible open set, it is easy to see by the same reasoning (using LES of pairs to get isomorphisms $H^*(X, A_i) \cong H^*(X)$ in every positive dimension), that
we have the following commutative diagram:

\[
\begin{array}{ccc}
H^{k_1}(X, A_1; R) \times \ldots \times H^{k_n}(X, A_n; R) & \cong & H^{k_1+\ldots+k_n}(X, A_1 \cup \ldots \cup A_n; R) \\
\downarrow \cong & & \downarrow \\
H^{k_1}(X; R) \times \ldots \times H^{k_n}(X; R) & \cong & H^{k_1+\ldots+k_n}(X; R)
\end{array}
\]

The upper-right-hand term is 0, so the cup product in the bottom row is 0 (when all \(k_i > 0\)).

3.2.11. Let \(f : S^{k+l} \to S^k \times S^l\) and \(p_1, p_2\) be the projection maps of \(S^k \times S^l\) onto the first and second factor respectively. Künneth formula tells us that \(H^{k+l}(S^k \times S^l; \mathbb{Z}) \cong \mathbb{Z}\) and is generated by \(c = p^*(a) \cup p^*(b)\), where \(a \in H^k(S^k; \mathbb{Z})\), \(b \in H^l(S^l; \mathbb{Z})\) are generators. However, since \(H^k(S^{k+l}; \mathbb{Z})\) and \(H^l(S^{k+l}; \mathbb{Z})\) are both zero, by naturality of cup product,

\[
f^*(c) = f^*(p^*(a) \cup p^*(b)) = f(p^*(a)) \cup f(p^*(b)) = 0
\]

Hence, \(f\) induces the zero map on \(H^{k+l}(-, \mathbb{Z})\). By naturality of universal coefficient theorem, it means the dual of the map \(f_* : H_{k+l}(S^{k+l}; \mathbb{Z}) \cong \mathbb{Z} \to H_{k+l}(S^k \times S^l; \mathbb{Z}) \cong \mathbb{Z}\) is zero, and hence \(f_*\) is trivial on the only non-trivial positive dimensional homology group of \(S^{k+l}\). The result follows.

3.2.15. Coefficient of \(t^k\) in \(p(X \times Y)\), by Künneth formula, is

\[
\dim_F H^k(X \times Y) = \dim_F \bigoplus_{i+j=k} H^i(X) \otimes H^j(Y)
\]

\[
= \sum_{i+j=k} \dim_F H^i(X) \otimes H^j(Y)
\]

which is the coefficient of \(t^k\) in \(p(X)p(Y)\). Hence, \(p(X \times Y) = p(X)p(Y)\).

The cohomology groups of \(S^n\) with field coefficients are \(H^0(S^n; F) = H^n(S^n; F) = F\) and 0 otherwise (say, by cellular cohomology), so \(p(S^n) = 1 + t^n\) for any field \(F\).

If \(\text{char } F \neq 2\), universal coefficients theorem for cohomology gives us that \(H^i(RP^n; F) \cong F\) if \(i = 0\) or \(n\) odd, but is 0 otherwise. So \(p(RP^n) = 1 + t^n\) if \(n\) is odd, \(p(RP^n) = 1\) if \(n\) is even. If \(\text{char } F = 2\), then since all maps in the cell structure are 0, every cohomology group up to dimension \(n\) is \(F\), so \(p(RP^n) = \sum_{i=0}^n t^i\). Similarly, for \(RP^\infty\), if \(\text{char } F \neq 2\), \(p(RP^\infty) = 1\), while if \(\text{char } F = 2\), \(p(RP^\infty) = \sum_{i=0}^\infty t^i\).

Cellular cohomology shows that for any field \(F\), \(H^{2i}(CP^n; F) \cong F\) for \(0 \leq i \leq n\), and is 0 in odd degree. So \(p(CP^n) = \sum_{i=0}^{2n} t^{2i}\). Similarly, the cell structure on \(CP^\infty\) implies that \(H^i(CP^\infty; F) \cong F\) for all \(i\) even, so \(p(CP^n) = \sum_{i=0}^\infty t^{2i}\).

For \(S^3 \times CP^\infty\), we use the product formula shown above to get:

\[
p(S^3 \times CP^\infty) = p(S^3)p(CP^\infty) = (1 + t^3)(1 + t^2 + t^4 + \ldots) = 1 + \sum_{i=2}^{\infty} t^i
\]
For $\mathbb{CP}^\infty/\mathbb{CP}^1$, we use the long exact sequence of a pair (and the cellular structures of these two spaces to see what the inclusion does) to conclude that

$$p(\mathbb{CP}^\infty/\mathbb{CP}^1) = 1 + \sum_{i=2}^{\infty} t^{2i}$$

For $S^6 \times \mathbb{HP}^\infty$ we use the product formula shown above to get:

$$p(S^6 \times \mathbb{HP}^\infty) = p(S^6)p(\mathbb{HP}^\infty) = (1 + t^6)(1 + t^4 + t^8 + \ldots) = 1 + \sum_{i=2}^{\infty} t^{2i}$$

Note that $S^6 \times \mathbb{CP}^\infty$ and $\mathbb{CP}^\infty/\mathbb{CP}^1$ have isomorphic cohomology groups in all degrees - their Poincare series are the same. But the cohomology rings of these two spaces are different - cup product structures detect the difference - persuade yourself!

3.B.1 For simplicity, let us first assume $m, n$ are both even. The non-zero part of the cellular chain complexes of $\mathbb{RP}^m, \mathbb{RP}^n$ with respect to $\mathbb{Z}$-coefficient both have the from

$$\mathbb{Z} \overset{2}{\rightarrow} \mathbb{Z} \overset{0}{\rightarrow} \mathbb{Z} \overset{2}{\rightarrow} \mathbb{Z} \overset{n}{\rightarrow} \mathbb{Z} \overset{0}{\rightarrow} \mathbb{Z} \overset{2}{\rightarrow} \mathbb{Z} \overset{n}{\rightarrow} \mathbb{Z} \overset{0}{\rightarrow} \mathbb{Z} \overset{2}{\rightarrow} \mathbb{Z} \overset{n}{\rightarrow} \mathbb{Z}$$

Taking their tensor product, we obtain

A careful observation reveals that the double complex can be thought as the direct sum of the leftmost column and the bottom row, and a bunch of squares of the form

$$\mathbb{Z} \overset{2}{\rightarrow} \mathbb{Z} \overset{0}{\rightarrow} \mathbb{Z} \overset{2}{\rightarrow} \mathbb{Z} \overset{0}{\rightarrow} \mathbb{Z}$$

since any arrows between them are zero. Hence, the homology group of the double complex is the direct sum of the homology of them. At homology level, any such square contributes to a $\mathbb{Z}_2$ at the bottom left corner, and another $\mathbb{Z}_2$ at the diagonal corresponding to odd dimension. The homology groups contributed
by the leftmost column and bottom row can be calculated easily. Pictorially, the homology of the double
complex can be represented as

\[
\begin{array}{cccc}
0 & Z_2 & 0 & \cdots & Z_2 & 0 \\
Z_2 & Z_2 & 0 & \cdots & Z_2 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & Z_2 & 0 & \cdots & Z_2 & 0 \\
Z_2 & Z_2 & 0 & \cdots & Z_2 & 0 \\
Z_2 & Z_2 & 0 & \cdots & Z_2 & 0 \\
\end{array}
\]

and the homology group \( H_k(\mathbb{RP}^m \times \mathbb{RP}^n; \mathbb{Z}) \) is equal to the direct sum of the groups at the \((i, j)\) position, \( i + j = k \).

The cases where \( m, n \) may be odd is very similar, except that we need to add another column to the
right or a row on the top in the double complex. In such cases, the homology is the direct sum of those
above in the case of \( m, n \) were even, plus that of the extra row or/and column. For instance, if \( n \) is odd
and \( m \) is even, we get

\[
\begin{array}{cccc}
Z & Z_2 & 0 & \cdots & Z_2 & 0 \\
0 & Z_2 & 0 & \cdots & Z_2 & 0 \\
Z_2 & Z_2 & 0 & \cdots & Z_2 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & Z_2 & 0 & \cdots & Z_2 & 0 \\
Z_2 & Z_2 & 0 & \cdots & Z_2 & 0 \\
Z & Z_2 & 0 & \cdots & Z_2 & 0 \\
Z_2 & Z_2 & 0 & \cdots & Z_2 & 0 \\
\end{array}
\]

For the case of \( \mathbb{Z}_2 \)-coefficient, things are much simpler. All the \( Z \) in the double complex are replaced by
\( \mathbb{Z}_2 \) and all the differential become zero. Hence, the homology group \( H_k(\mathbb{RP}^m \times \mathbb{RP}^n; \mathbb{Z}_2) \) is the direct sum
of the corresponding \((i, j)\) in

\[
\begin{array}{cccc}
Z_2 & Z_2 & \cdots & Z_2 \\
Z_2 & Z_2 & \cdots & Z_2 \\
\vdots & \vdots & \ddots & \vdots \\
Z_2 & Z_2 & \cdots & Z_2 \\
\end{array}
\]

Therefore,

\[
H_k(\mathbb{RP}^m \times \mathbb{RP}^n; \mathbb{Z}_2) = \bigoplus_{0 \leq i \leq m, \atop 0 \leq j \leq n} \mathbb{Z}_2
\]
Cohomology can be calculated similar and we skip the details here.

3.B.3 At cellular chain level, \( f : M(\mathbb{Z}_m, n) \to S^{n+1} \) is represented by

\[
\begin{array}{cccccccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & 0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathbb{Z} & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
& & & & & & & & \\
\end{array}
\]

respectively. Here, the leftmost terms denote \( C_{n+2} \).

Taking the product of it with the identity chain map on the cellular chain of \( M(\mathbb{Z}_m, n) \), we obtain the cellular chain map of \( f \times 1 : M(\mathbb{Z}_m, n) \times M(\mathbb{Z}_m, n) \to S^{n+1} \times M(\mathbb{Z}_m, n) \)

\[
\begin{array}{cccccccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{Z} & \rightarrow & \cdots \\
\end{array}
\]

where the leftmost terms are \( C_{2n+3} \). Calculating the homology we see the \( H_{2n+1} \) of both product spaces are \( \mathbb{Z}_m \) and \( (f \times 1)_* \) is an isomorphism between them.

Next, we look at the Künneth formula.

\[
\begin{array}{cccccccc}
0 & \rightarrow & 0 & \rightarrow & H_{2n+1}(M(\mathbb{Z}_m, n) \times M(\mathbb{Z}_m, n)) & \rightarrow & \text{Tor}(\mathbb{Z}_m, \mathbb{Z}_m) & \rightarrow & 0 \\
\downarrow & & \downarrow & & (f \times 1)_* & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathbb{Z} \otimes \mathbb{Z}_m & \rightarrow & H_{2n+1}(S^{n+1} \times M(\mathbb{Z}_m, n)) & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

If the splitting is natural, then there will be an isomorphism \( \phi \) such that

\[
\begin{array}{cccccccc}
H_{2n+1}(M(\mathbb{Z}_m, n) \times M(\mathbb{Z}_m, n)) & \leftarrow & \phi & \rightarrow & \text{Tor}(\mathbb{Z}_m, \mathbb{Z}_m) & \rightarrow & 0 \\
\downarrow & & (f \times 1)_* & & \downarrow & & \downarrow \\
H_{2n+1}(S^{n+1} \times M(\mathbb{Z}_m, n)) & \leftarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

is commutative. It is impossible because this would imply the automorphism \((f \times 1)_* \circ \phi \) of \( \mathbb{Z}_m \) can be factored through zero.