Instructions. This is a take-home examination. Write out complete solutions to the following problems. You may use your textbooks, class notes, and may use or quote any result discussed in class or in the book. However you may not consult any other books or materials. You must work alone on this examination. The exam is due on Monday, May 15 in class. No late examinations will be accepted. Good luck!!

1. a. Show that for a degree 1 map $f : M \to N$ of connected closed orientable manifolds, the induced map $f_* : \pi_1 M \to \pi_1 N$ is surjective, hence also $f_* : H_1(M) \to H_1(N)$. (Hint. Lift $f$ to the covering space $\tilde{N} \to N$ corresponding to the subgroup $\text{Im } f_* \subset \pi_1 N$. Consider the two cases that this covering is finite-sheeted or infinite-sheeted.)

b. If $M_g$ denotes the closed orientable surface of genus $g$, show that degree 1 maps $M_g \to M_k$ exist iff $g \geq k$.

2. a. Show that a nonsingular symmetric or skew-symmetric bilinear pairing over a field $F$, of the form $F^n \times F^n \to F$, cannot be identically zero when restricted to all pairs of vectors, $v, w$ in a $k$-dimensional subspace $V \subset F^n$, if $k > n/2$.

b. Use part a. to show that if the closed orientable surface $\Sigma_g$ of genus $g$ retracts onto a graph $X \subset \Sigma_g$, then $H_1(X)$ has rank at most $g$. 
3. In this problem we describe a generalization of the Poincare duality theorem, and your goal is to prove some consequences of it.

Let $M$ be an oriented $n$-manifold. Let $K \subset L \subset M$ be CW-complexes that are embedded as compact subsets of $M$. An argument similar to the proof of Poincare duality proves the following generalization (known as Poincare-Lefschetz duality):

**Theorem 1** There is an isomorphism,

$$ D : H^p(K, L; \mathbb{Z}) \xrightarrow{\cong} H_{n-p}(M - L, M - K; \mathbb{Z}). $$

Use this theorem to prove the following results:

a. (Alexander Duality) Prove that if $A \subset \mathbb{R}^n$ is a compact CW-complex, then

$$ H_q(\mathbb{R}^n - A; \mathbb{Z}) \cong H^{n-q-1}(A; \mathbb{Z}). $$

b. Let $K \subset S^3$ be a knot. That is, $K$ is the image of a smooth embedding $e : S^1 \subset S^3$. Let $\pi$ be the fundamental group of the complement,

$$ \pi = \pi_1(S^3 - K). $$

Let $[\pi, \pi] \subset \pi$ be the commutator subgroup. Prove that the abelianization $\pi/[[\pi, \pi]$ is isomorphic to $\mathbb{Z}$.

c. Let $M$ is a connected, orientable, compact $n$-manifold with $H_1(M; \mathbb{Z}) = 0$. Let $A \subset M$ be a compact CW-complex. Prove that $H^{n-1}(A; \mathbb{Z})$ is a free abelian group whose rank is one less than the number of connected components of the complement $M - A$.

d. Show that $\mathbb{R}P^2$ does not embed in $\mathbb{R}^3$.

e. Consider the function

$$ f : \mathbb{R}P^2 \to \mathbb{R}^4 $$

$$ [x_1, x_2, x_3] \to (x_2x_3, x_1x_3, x_1x_2, x_1^2 + 2x_2^2 + 3x_3^2). $$

(We are describing points in $\mathbb{R}P^2$ by unit vectors $(x_1, x_2, x_3) \in \mathbb{R}^3$, subject to the equivalence relation $(x_1, x_2, x_3) \sim -(x_1, x_2, x_3)$.) Show that $f$ is a smooth embedding.
4. Let $N \xrightarrow{\iota} M \xrightarrow{e_M} \mathbb{R}^K$ be smooth embeddings, with $N$ and $M$ compact. Let $e_N : N \hookrightarrow \mathbb{R}^K$ be the composite embedding $e_N = e_M \circ \iota$. Define the normal bundle of the embedding $\iota$ by

$$\nu_\iota = \{(x, v) \in N \times \mathbb{R}^K : v \in T_x M \text{ and } v \perp T_x N\}.$$ 

Consider the zero section $z : N \hookrightarrow \nu_\iota$ defined by $z(x) = (x, 0)$. Prove the following version of the tubular neighborhood theorem.

**Theorem 2** The embedding $\iota : N \hookrightarrow M$ extends to an embedding $\tilde{\iota} : \nu_\iota \hookrightarrow M$, which is a diffeomorphism onto an open subspace of $M$. By $\tilde{\iota}$ extending $\iota$, we mean that $\tilde{\iota} \circ z = \iota : N \hookrightarrow \nu_\iota \hookrightarrow M$.

**Hint.** Consider tubular neighborhoods $(V^M_\rho, r_M)$ and $(V^N_\rho, r_N)$ of the embeddings $e_M$ and $e_N$ respectively, as in theorem 9.2 of Madsen and Tornehave. Assume both tubular neighborhoods have constant radius $= \rho > 0$, which we can do since $M$ and $N$ are assumed to be compact. Show that every element $y \in V^N_\rho$ has a unique representation as

$$y = r_N(y) + u_y + v_y \in \mathbb{R}^K$$

where $u_y \in T_{r_N(y)}M$ and $v_y \in (T_{r_N(y)}M)^\perp$.

5. Let $M^n$ be a Riemannian manifold (i.e a smooth manifold endowed with a Riemannian metric, or equivalently a Euclidean structure on its tangent bundle). Let $f : M \to \mathbb{R}$ be a smooth map. Define the gradient vector field $\nabla f$ by demanding that $(\nabla f)_p \in T_p M$ satisfies

$$(\langle \nabla f)_p, v \rangle = df_p(v)$$

for all $v \in T_p M$.

a. For $W$ an open subset of $\mathbb{R}^n$, let $h : W \to M$, be a local chart. Show that we have

$$(\nabla f)_{h(x)} = \sum_{j=1}^n a_j(x) \frac{\partial}{\partial x_j}$$
where \( a_j \in C^\infty(W, \mathbb{R}) \), \( 1 \leq j \leq n \) is determined by the set of linear equations

\[
\sum_{i=1}^{n} g_{i,j}(x)a_j(x) = \frac{\partial f}{\partial x_i}(x) \quad (1 \leq i \leq n).
\]

Here the functions \( g_{i,j} : W \to \mathbb{R} \) are defined by the metric and the chart \( h \):

\[
g_{i,j}(x) = \langle (Dh)_x(e_i), (Dh)_x(e_j) \rangle_{h(x)}, \quad 1 \leq i, j \leq n,
\]

where \( e_1, \ldots, e_n \) is the standard basis of \( \mathbb{R}^n \). These functions are called the coefficients of the first fundamental form of the metric.

b. Show that the map \( \nabla f : M \to TM \) is smooth.

c. Let \( p \in M \) with \( (\nabla f)_p \neq 0 \). Set \( c = f(p) \). Show that \( f^{-1}(c) \) in a neighborhood of \( p \) is an \( (n-1) \)-dimensional smooth submanifold, and that \( (\nabla f)_p \) is a normal vector to \( f^{-1}(c) \) at \( p \).