Math 206 Final Exam

Name:

- You have 2 hours and 50 minutes to complete this exam.
- No notes, books, calculators, or other references are allowed.
- You must show all work to receive credit. Answers for which no work is shown will receive no credit (unless specifically stated otherwise).
- There are problems on the front and back of each page. Make sure that you answer all of the questions. There is extra paper available if you need it.
- Good luck! Have fun! Eat candy as necessary.

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1. (a) (8 points) Suppose that $A$ is an $n \times n$ matrix. We gave at least 20 statements that are equivalent to the statement “$A$ is invertible.” List 8 of these conditions. One of your conditions must be a criterion for invertibility that involves the eigenvalues of $A$, and one must be a criterion for invertibility that involves the determinant of $A$. 
(b) (4 points) Use one of your conditions from (a) to determine whether the matrix

\[ A = \begin{bmatrix}
1 & 3 & -2 \\
2 & 5 & -3 \\
-3 & 2 & -4
\end{bmatrix} \]

is invertible.
2. The \textit{augmented} matrices of three linear systems are given below in reduced row echelon form.

- System 1: \( Ax = b \),
  \[
  \begin{bmatrix}
  1 & 0 & 0 & | & 4 \\
  0 & 1 & 0 & | & 7 \\
  0 & 0 & 1 & | & -1 \\
  \end{bmatrix}
  \]

- System 2: \( By = c \),
  \[
  \begin{bmatrix}
  0 & 1 & -2 & 0 & | & 1 \\
  0 & 0 & 0 & 1 & | & 3 \\
  0 & 0 & 0 & 0 & | & 0 \\
  0 & 0 & 0 & 0 & | & 0 \\
  \end{bmatrix}
  \]

- System 3: \( Cz = d \),
  \[
  \begin{bmatrix}
  1 & 0 & 0 & | & 0 \\
  0 & 1 & 2 & | & 0 \\
  0 & 0 & 0 & | & 1 \\
  \end{bmatrix}
  \]

(a) (3 points) Which of the systems are consistent? Which are inconsistent?

(b) (3 points) How many solutions does each system have?

(c) (3 points) Which of the matrices \( A, B, C \) are invertible?
(d) (3 points) Find the rank of each matrix $A, B, C$.

(e) (3 points) Which of the matrices $A, B, C$ have linearly independent columns?

3. (3 points) Suppose that $X$ and $Y$ are invertible $3 \times 3$ matrices such that $\det(X) = 7$ and $\det(Y) = 10$. Find $\det \left( 2(YX)^T (XY)^{-1} \right)$.
4. (a) (2 points) If \( A \) is a nonzero \( 5 \times 3 \) matrix, what is the maximum possible value of \( \text{rank}(A) \)?

(b) (2 points) Give an example of 4 linearly independent vectors in \( \mathbb{R}^3 \), or state that it is not possible to do so.

(c) (2 points) Give an example of 4 vectors that span \( \mathbb{R}^3 \), or state that it is not possible to do so.
5. (a) (4 points) Suppose that \( \mathbf{u} \) is a vector in \( \mathbb{R}^n \) such that \( \mathbf{u} \cdot \mathbf{u} = 1 \). Let \( P = \mathbf{uu}^T \) and \( Q = I_n - 2P \). Show that \( Q^2 = I_n \). Note: make sure that you completely justify all of your work on this problem. In particular, if you use a theorem or property discussed in class, make sure you state where and how you are using it.
(b) (3 points) Find all possible eigenvalues of \( Q \), where \( Q \) is the matrix from part (a). Note: even if you were not able to solve (a), you may use the fact that \( Q^2 = I_n \) to solve part (b).
6. Let
\[
A = \begin{bmatrix}
1 & 3 & 3 \\
-3 & -5 & -3 \\
3 & 3 & 1
\end{bmatrix}
\text{ and } B = \begin{bmatrix}
2 & 4 & 3 \\
-4 & -6 & -3 \\
3 & 3 & 1
\end{bmatrix}.
\]
For this problem, you may use the fact that both matrices have the same characteristic polynomial: \(p_A(\lambda) = p_B(\lambda) = -(\lambda - 1)(\lambda + 2)^2\).

(a) (4 points) Find all eigenvectors of \(A\).

(b) (4 points) Find all eigenvectors of \(B\).
(c) (2 points) Which matrix \( A \) or \( B \) is diagonalizable?

(d) (5 points) Diagonalize the matrix stated in (d), i.e. find an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( A = PDP^{-1} \) or \( B = PDP^{-1} \).
7. (8 points) Suppose that $A$ is an $n \times n$ matrix such that all eigenvalues of $A$ are positive. Explain why the matrix $A + I_n$ must be invertible.
8. Let $A$ be an $m \times n$ orthogonal matrix, and let $x$ and $y$ be vectors in $\mathbb{R}^n$.

(a) (5 points) Show that $(Ax) \cdot (Ay) = x \cdot y$.

(b) (5 points) Use (a) to show that $||Ax|| = ||x||$. 

9. (8 points) Let \( \mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \) and \( \mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \). Write \( \mathbf{y} \) as a sum of two orthogonal vectors, one in Span(\( \mathbf{u} \)) and one orthogonal to \( \mathbf{u} \).
10. Suppose that \( \{u_1, u_2, u_3\} \) is an orthogonal set of vectors in \( \mathbb{R}^4 \), and \( ||u_1|| = 2 \), \( ||u_2|| = 3 \), and \( ||u_3|| = 4 \). Let \( y = 2u_1 - 5u_2 + u_3 \).

(a) (4 points) Find \( ||y|| \).

(b) (4 points) Find \( y \cdot u_1 \).
11. (8 points) Determine whether each of the following statements is True or False. These questions will be scored as follows: +1 points for a correct answer, 0 points for no response, and \(-1/2\) points for an incorrect answer, with a minimum possible total score of 0.

(a) The only possible eigenvalue of an orthogonal matrix is \(\lambda = 1\).

(b) If \(A\) is an \(n \times n\) matrix such that \(\text{rref}(A) = I_n\), then \(\det(A) = 1\).

(c) If \(A\) and \(B\) are orthogonal matrices, then \(AB\) is also an orthogonal matrix.

(d) Every orthogonal matrix is invertible.

(e) It’s possible to find 10 linearly independent vectors in \(\mathbb{R}^{11}\).

(f) It’s possible to find 11 linearly independent vectors in \(\mathbb{R}^{10}\).

(g) If \(A\) is a \(4 \times 4\) matrix with eigenvalues \(0,1,2,3\), then \(\text{rank}(A) = 4\).

(h) If \(A\) and \(B\) are similar matrices, then \(A^2\) and \(B^2\) are also similar.