Math 206 Exam 2
Tuesday, November 17, 2009

Name:

- You have 50 minutes to complete this exam.
- No notes, books, calculators, or other references are allowed.
- You must show all work to receive credit. Answers for which no work is shown will receive no credit (unless specifically stated otherwise).
- There are problems on the front and back of each page. Make sure that you answer all of the questions. There is extra paper available if you need it.
- Good luck! Have fun! Eat candy as necessary.

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1. (6 points) Let

\[ A = \begin{bmatrix} 2 & 0 & 10 \\ 0 & 7 + x & -3 \\ 0 & 4 & x \end{bmatrix}. \]

Find all values of \( x \) such that \( A \) is invertible. Make sure that you completely justify your answer.
2. (5 points total) Consider the matrix

\[ A = \begin{bmatrix}
1 & 5 & 4 & 3 & 2 \\
1 & 6 & 6 & 6 & 6 \\
1 & 7 & 8 & 10 & 12 \\
1 & 6 & 6 & 7 & 8 \\
\end{bmatrix}. \]

You may use the fact that

\[ \text{rref}(A) = \begin{bmatrix}
1 & 0 & -6 & 0 & 6 \\
0 & 1 & 2 & 0 & -2 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}. \]

(a) (3 points) Find a basis for the nullspace of \( A \).

(b) (2 points) Find a non-zero vector that is not one of the columns of \( A \) in the column space of \( A \). Make sure that you clearly explain why the vector you provide is in the column space of \( A \).
3. (4 points) Suppose that $A$ is a $5 \times 8$ matrix such that

\[
\begin{align*}
\begin{bmatrix} 7 \\ 0 \\ 4 \\ 1 \\ 2 \end{bmatrix}, & \quad \begin{bmatrix} -3 \\ 1 \\ 2 \\ -6 \\ 8 \end{bmatrix}, & \quad \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \\ 1 \end{bmatrix}
\end{align*}
\]

is a basis for the column space of $A$. Find $p$ and $q$ so that the following statement is true: The nullspace of $A$ is a $p$-dimensional subspace of $\mathbb{R}^q$. 
4. (8 points total) Consider the matrix

\[ A = \begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix}. \]

(a) (2 points) Find all eigenvalues of \( A \).

(b) (6 points) Find a basis for each eigenspace of \( A \).
5. (6 points) Suppose that $\lambda$ is an eigenvalue of an invertible matrix $A$ with corresponding eigenvector $\mathbf{v}$. Determine whether $\mathbf{v}$ is an eigenvector of the matrix $A + cI_n$, where $c$ is a scalar. If so, what is the corresponding eigenvalue?
6. (8 points total) The trace of an $n \times n$ matrix $A$ is the sum of the entries on the main diagonal of the matrix, and is denoted $\text{tr}(A)$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) (4 points) Show that the characteristic polynomial of $A$ is

$$p(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A).$$

(b) (4 points) Suppose that $A$ has two distinct, real eigenvalues $\lambda_1$ and $\lambda_2$. Show that

$$\text{tr}(A) = \lambda_1 + \lambda_2.$$
7. (8 points) Suppose that \[
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}
\] is a basis for the nullspace of the matrix \( A + 5I_3 \) and that \[
\begin{pmatrix}
2 \\
0 \\
4
\end{pmatrix}
\] is a basis for the nullspace of the matrix \( A - 2I_3 \). Find \( A^2 \begin{pmatrix}
3 \\
-2 \\
5
\end{pmatrix} \).
8. (5 points) Determine whether each of the following statements is True or False. These questions will be scored as follows: +1 points for a correct answer, 0 points for no response, and \(-1/2\) points for an incorrect answer.

(a) If \(A\) and \(B\) are \(n \times n\) matrices, and \(P\) is an invertible \(n \times n\) matrix such that \(A = PBP^{-1}\), then \(\det(A) = \det(B)\).

(b) If the characteristic polynomial of an \(n \times n\) matrix \(A\) is \(p(\lambda) = (\lambda - 1)^n + 2\), then \(A\) is invertible.

(c) If \(A^2\) is an invertible \(n \times n\) matrix, then \(A^3\) is also invertible.

(d) If \(A\) is a \(3 \times 3\) matrix such that \(\det(A) = 7\), then \(\det(2A^T A^{-1}) = 2\).

(e) If \(v\) is an eigenvector of an \(n \times n\) matrix \(A\) with corresponding eigenvalue \(\lambda_1\), and if \(w\) is an eigenvector of \(A\) with corresponding eigenvalue \(\lambda_2\), then \(v + w\) is an eigenvector of \(A\) with corresponding eigenvalue \(\lambda_1 + \lambda_2\).