1. Prove that for each positive integer \( n \), there are pairwise relatively prime integers \( k_0, k_1, \ldots, k_n \), all strictly greater than 1, such that \( k_0k_1 \cdots k_n - 1 \) is the product of two consecutive integers.

**First solution:** We proceed by induction. The case \( n = 1 \) is clear, since we may pick \( k_0 = 3 \) and \( k_1 = 7 \).

Let us assume now that for a certain \( n \) there are pairwise relatively prime integers \( 1 < k_0 < k_1 < \cdots < k_n \) such that \( k_0k_1 \cdots k_n - 1 = a_n(a_n - 1) \), for some positive integer \( a_n \). Then choosing \( k_{n+1} = a_n^2 + a_n + 1 \) yields

\[
k_0k_1 \cdots k_{n+1} = (a_n^2 - a_n + 1)(a_n^2 + a_n + 1) = a_n^4 + a_n^2 + 1,
\]

so \( k_0k_1 \cdots k_{n+1} - 1 \) is the product of the two consecutive integers \( a_n^2 \) and \( a_n^2 + 1 \). Moreover,

\[
\gcd(k_0k_1 \cdots k_n, k_{n+1}) = \gcd(a_n^2 - a_n + 1, a_n^2 + a_n + 1) = 1,
\]

hence \( k_0, k_1, \ldots, k_{n+1} \) are pairwise relatively prime. This completes the proof.

**Second solution:** Write the relation to be proved as

\[
4k_0k_1 \cdots k_n = 4a(a + 1) + 4 = (2a + 1)^2 + 3.
\]

There are infinitely many primes for which \(-3\) is a quadratic residue. Let \( 2 < p_0 < p_1 < \cdots < p_n \) be such primes. Using the Chinese Remainder Theorem to specify \( a \) modulo \( p_n \), we can find an integer \( a \) such that \((2a+1)^2 + 3 = 4p_0p_1 \cdots p_nm\) for some positive integer \( m \).

Grouping the factors of \( m \) appropriately with the \( p_i \)'s, we obtain \((2a+1)^2 + 3 = 4k_0k_1 \cdots k_n\) with \( k_i \) pairwise relatively prime. We then have \( k_0k_1 \cdots k_n - 1 = a(a + 1) \), as desired.

**Third solution:** We are supposed to show that for every positive integer \( n \), there is a positive integer \( x \) such that \( x(x + 1) + 1 = x^2 + x + 1 \) has at least \( n \) distinct prime divisors. We can actually prove a more general statement.

**Claim.** Let \( P(x) = a_dx^d + \cdots + a_1x + 1 \) be a polynomial of degree \( d \geq 1 \) with integer coefficients. Then for every positive integer \( n \), there is a positive integer \( x \) such that \( P(x) \) has at least \( n \) distinct prime divisors.
The proof follows from the following two lemmas.

**Lemma 1.** The set

\[ Q = \{ p | p \text{ a prime for which there is an integer } x \text{ such that } p \text{ divides } P(x) \} \]

is infinite.

**Proof.** The proof is analogous to Euclid’s proof that there are infinitely many primes. Namely, if we assume that there are only finitely many primes \( p_1, p_2, \ldots, p_k \) in \( Q \), then for each integer \( m \), \( P(mp_1p_2 \cdots p_k) \) is an integer with no prime factors, which must equal 1 or \(-1\). However, since \( P \) has degree \( d \geq 1 \), it takes each of the values 1 and \(-1\) at most \( d \) times, a contradiction. \( \square \)

**Lemma 2.** Let \( p_1, p_2, \ldots, p_n, \ n \geq 1 \) be primes in \( Q \). Then there is a positive integer \( x \) such that \( P(x) \) is divisible by \( p_1p_2 \cdots p_n \).

**Proof.** For \( i = 1, 2, \ldots, n \), since \( p_i \in Q \) we can find an integer \( c_i \) such that \( P(x) \) is divisible by \( p_i \) whenever \( x \equiv c_i \pmod{p_i} \). By the Chinese Remainder Theorem, the system of \( n \) congruences \( x \equiv c_i \pmod{p_i}, i = 1, 2, \ldots, n \) has positive integer solutions. For every positive integer \( x \) that solves this system, \( P(x) \) is divisible by \( p_1p_2 \cdots p_n \). \( \square \)

This problem was suggested by Titu Andreescu.

2. Let \( ABC \) be an acute, scalene triangle, and let \( M, N \), and \( P \) be the midpoints of \( BC, CA \), and \( AB \), respectively. Let the perpendicular bisectors of \( AB \) and \( AC \) intersect ray \( AM \) in points \( D \) and \( E \) respectively, and let lines \( BD \) and \( CE \) intersect in point \( F \), inside of triangle \( ABC \). Prove that points \( A, N, F \), and \( P \) all lie on one circle.

**First solution:** Let \( O \) be the circumcenter of triangle \( ABC \). We prove that

\[ \angle APO = \angleANO = \angle AFO = 90^\circ. \] (1)

It will then follow that \( A, P, O, F, N \) lie on the circle with diameter \( AO \). Indeed, the fact that the first two angles in (1) are right is immediate because \( OP \) and \( ON \) are the perpendicular bisectors of \( AB \) and \( AC \), respectively. Thus we need only prove that \( \angle AFO = 90^\circ \).
We may assume, without loss of generality, that $AB > AC$. This leads to configurations similar to the ones shown above. The proof can be adapted to other configurations. Because $PO$ is the perpendicular bisector of $AB$, it follows that triangle $ADB$ is an isosceles triangle with $AD = BD$. Likewise, triangle $AEC$ is isosceles with $AE = CE$. Let $x = \angle ABD = \angle BAD$ and $y = \angle CAE = \angle ACE$, so $x + y = \angle BAC$.  

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Applying the Law of Sines to triangles $ABM$ and $ACM$ gives
\[
\frac{BM}{\sin x} = \frac{AB}{\sin \angle BMA} \quad \text{and} \quad \frac{CM}{\sin y} = \frac{AC}{\sin \angle CMA}.
\]
Taking the quotient of the two equations and noting that $\sin \angle BMA = \sin \angle CMA$ we find
\[
\frac{BM \sin y}{CM \sin x} = \frac{AB \sin \angle CMA}{AC \sin \angle BMA} = \frac{AB}{AC}.
\]
Because $BM = MC$, we have
\[
\frac{\sin x}{\sin y} = \frac{AC}{AB}. \tag{2}
\]
Applying the law of sines to triangles $ABF$ and $ACF$ we find
\[
\frac{AF}{\sin x} = \frac{AB}{\sin \angle AFB} \quad \text{and} \quad \frac{AF}{\sin y} = \frac{AC}{\sin \angle AFC}.
\]
Taking the quotient of the two equations yields
\[
\frac{\sin x}{\sin y} = \frac{AC \sin \angle AFB}{AB \sin \angle AFC}, \quad \text{so by (2),} \quad \sin \angle AFB = \sin \angle AFC. \tag{3}
\]
Because $\angle ADF$ is an exterior angle to triangle $ADB$, we have $\angle EDF = 2x$. Similarly, $\angle DEF = 2y$. Hence
\[
\angle EFD = 180^\circ - 2x - 2y = 180^\circ - 2\angle BAC.
\]
Thus $\angle BFC = 2\angle BAC = \angle BOC$, so $BOFC$ is cyclic. In addition,
\[
\angle AFB + \angle AFC = 360^\circ - 2\angle BAC > 180^\circ,
\]
and hence, from (3), $\angle AFB = \angle AFC = 180^\circ - \angle BAC$. Because $BOFC$ is cyclic and $\triangle BOC$ is isosceles with vertex angle $\angle BOC = 2\angle BAC$, we have $\angle OFB = \angle OCB = 90^\circ - \angle BAC$. Therefore,
\[
\angle AFO = \angle AFB - \angle OFB = (180^\circ - \angle BAC) - (90^\circ - \angle BAC) = 90^\circ.
\]
This completes the proof. \qed

**Second solution:** Invert the figure about a circle centered at $A$, and let $X'$ denote the image of the point $X$ under this inversion. Find point $F_1'$ so that $AB'F_1'C'$ is a parallelogram and let $Z'$ denote the center of this parallelogram. Note that $\triangle BAC \sim$
\(\triangle C'AB'\) and \(\triangle BAD \sim \triangle D'AB'\). Because \(M\) is the midpoint of \(BC\) and \(Z'\) is the midpoint of \(B'C'\), we also have \(\triangle BAM \sim \triangle C'AZ'\). Thus

\[\angle AF'B' = \angle F'AC' = \angle Z'AC' = \angle MAB = \angle DAB = \angle DBA = \angle AD'B'.\]

Hence quadrilateral \(AB'D'F'\) is cyclic and, by a similar argument, quadrilateral \(AC'E'F'_1\) is also cyclic. Because the images under the inversion of lines \(BDF\) and \(CFE\) are circles that intersect in \(A\) and \(F'\), it follows that \(F'_1 = F''\).

Next note that \(B', Z',\) and \(C'\) are collinear and are the images of \(P', F',\) and \(N',\) respectively, under a homothety centered at \(A\) and with ratio \(1/2\). It follows that \(P', F'\) and \(N'\) are collinear, and then that the points \(A, P, F\) and \(N\) lie on a circle.

This problem was suggested by Zuming Feng. The second solution was contributed by Gabriel Carroll.

3. Let \(n\) be a positive integer. Denote by \(S_n\) the set of points \((x, y)\) with integer coordinates such that

\[|x| + \left|y + \frac{1}{2}\right| < n.\]

A path is a sequence of distinct points \((x_1, y_1), (x_2, y_2), \ldots, (x_\ell, y_\ell)\) in \(S_n\) such that, for \(i = 2, \ldots, \ell\), the distance between \((x_i, y_i)\) and \((x_{i-1}, y_{i-1})\) is 1 (in other words, the points \((x_i, y_i)\) and \((x_{i-1}, y_{i-1})\) are neighbors in the lattice of points with integer coordinates).
Prove that the points in \( S_n \) cannot be partitioned into fewer than \( n \) paths (a partition of \( S_n \) into \( m \) paths is a set \( \mathcal{P} \) of \( m \) nonempty paths such that each point in \( S_n \) appears in exactly one of the \( m \) paths in \( \mathcal{P} \)).

**Solution:** Color the points in \( S_n \) as follows (see Figure 1):

- if \( y \geq 0 \), color \((x, y)\) white if \( x + y - n \) is even and black if \( x + y - n \) is odd;
- if \( y < 0 \), color \((x, y)\) white if \( x + y - n \) is odd and black if \( x + y - n \) is even.

![Figure 1: Coloring of \( S_3 \)](image)

Consider a path \((x_1, y_1), (x_2, y_2), \ldots, (x_\ell, y_\ell)\) in \( S_n \). A pair of successive points \((x_{i-1}, y_{i-1})\) and \((x_i, y_i)\) in the path is called a pair of successive black points if both points in the pair are colored black.

Suppose now that the points of \( S_n \) are partitioned into \( m \) paths and the total number of successive pairs of black points in all paths is \( k \). By breaking the paths at each pair of successive black points, we obtain \( k + m \) paths in each of which the number of black points exceeds the number of white points by at most one. Therefore, the total number of black points in \( S_n \) cannot exceed the number of white points by more than \( k + m \). On the other hand, the total number of black points in \( S_n \) exceeds the total number of white points by exactly \( 2n \) (there is exactly one more black point in each row of \( S_n \)). Therefore,

\[
2n \leq k + m.
\]
There are exactly $n$ adjacent black points in $S_n$ (call two points in $S_n$ adjacent if their distance is 1), namely the pairs

$$(x, 0) \text{ and } (x, -1),$$

for $x = -n + 1, -n + 3, \ldots, n - 3, n - 1$. Therefore $k \leq n$ (the number of successive pairs of black points in the paths in the partition of $S_n$ cannot exceed the total number of adjacent pairs of black points in $S_n$) and we have $2n \leq k + m \leq n + m$, yielding

$$n \leq m.$$

This problem was suggested by Gabriel Carroll.

4. Let $P$ be a convex polygon with $n$ sides, $n \geq 3$. Any set of $n - 3$ diagonals of $P$ that do not intersect in the interior of the polygon determine a triangulation of $P$ into $n - 2$ triangles. If $P$ is regular and there is a triangulation of $P$ consisting of only isosceles triangles, find all the possible values of $n$.

**Solution:** The answer is $n = 2^{m+1} + 2^k$, where $m$ and $k$ are nonnegative integers. In other words, $n$ is either a power of 2 (when $m + 1 = k$) or the sum of two nonequal powers of 2 (with $1 = 2^0$ being considered as a power of 2).

We start with the following observation.

**Lemma.** Let $Q = Q_0Q_1\ldots Q_t$ be a convex polygon with $Q_0Q_1 = Q_1Q_2 = \cdots = Q_{t-1}Q_t$. Suppose that $Q$ is cyclic and its circumcenter does not lie in its interior. If there is a triangulation of $Q$ consisting only of isosceles triangles, then $t = 2^a$, where $a$ is a positive integer.

**Proof.** We call an arc minor if its arc measure is less than or equal to $180^\circ$. By the given conditions, points $Q_1, \ldots, Q_{t-1}$ lie on the minor arc $Q_0Q_t$ of the circumcircle, so none of the angles $Q_iQ_jQ_k$ ($0 \leq i < j < k \leq t$) is acute. (See the left-hand side diagram shown below.) It is not difficult to see that $Q_0Q_t$ is longer than each other side or diagonal of $Q$. Thus $Q_0Q_t$ must be the base of an isosceles triangle in the triangulation of $Q$. Therefore, $t$ must be even. We write $t = 2s$. Then $Q_0Q_sQ_t$ is an isosceles triangle in the triangulation. We can apply the same process to polygon $Q_0Q_1\ldots Q_s$ and show that $s$ is even. Repeating this process leads to the conclusion that $t = 2^a$ for some positive integer $a$. 

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The results of the lemma can be generalized by allowing \( a = 0 \) if we consider the degenerate case \( Q = Q_0Q_1 \).

We are ready to prove our main result. Let \( P = P_1P_2 \ldots P_n \) denote the regular polygon. There is an isosceles triangle in the triangulation such that the center of \( P \) lies within the boundary of the triangle. Without loss of generality, we may assume that \( P_1P_iP_j \), with \( P_1P_i = P_1P_j \) (that is, \( P_j = P_{n-i+2} \)), is this triangle. Applying the Lemma to the polygons \( P_1 \ldots P_i, P_i \ldots P_j \), and \( P_j \ldots P_1 \), we conclude that there are \( 2^m-1, 2^k-1, 2^m-1 \) (where \( m \) and \( k \) are nonnegative integers) vertices in the interiors of the minor arcs \( \widehat{P_1P_i}, \widehat{P_iP_j}, \widehat{P_jP_1} \), respectively. (In other words, \( i = 2^m + 1, j = 2^k + i \).) Hence

\[
n = 2^m - 1 + 2^k - 1 + 2^m - 1 + 3 = 2^{m+1} + 2^k,
\]

where \( m \) and \( k \) are nonnegative integers. The above discussion can easily lead to a triangulation consisting of only isosceles triangles for \( n = 2^{m+1} + 2^k \). (The middle diagram shown above illustrates the case \( n = 18 = 2^{3+1} + 2^1 \). The right-hand side diagram shown above illustrates the case \( n = 16 = 2^{2+1} + 2^3 \).)

This problem was suggested by Gregory Galperin.

5. Three nonnegative real numbers \( r_1, r_2, r_3 \) are written on a blackboard. These numbers have the property that there exist integers \( a_1, a_2, a_3 \), not all zero, satisfying \( a_1r_1 + a_2r_2 + a_3r_3 = 0 \). We are permitted to perform the following operation: find two numbers \( x, y \) on the blackboard with \( x \leq y \), then erase \( y \) and write \( y - x \) in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

**Solution:** If two of the \( a_i \) vanish, say \( a_2 \) and \( a_3 \), then \( r_1 \) must be zero and we are done. Assume at most one \( a_i \) vanishes. If any one \( a_i \) vanishes, say \( a_3 \), then \( r_2/r_1 = -a_1/a_2 \)
is a nonnegative rational number. Write this number in lowest terms as $p/q$, and put $r = r_2/p = r_1/q$. We can then write $r_1 = qr$ and $r_2 = pr$. Performing the Euclidean algorithm on $r_1$ and $r_2$ will ultimately leave $r$ and 0 on the blackboard. Thus we are done again.

Thus it suffices to consider the case where none of the $a_i$ vanishes. We may also assume none of the $r_i$ vanishes, as otherwise there is nothing to check. In this case we will show that we can perform an operation to obtain $r'_1, r'_2, r'_3$ for which either one of $r'_1, r'_2, r'_3$ vanishes, or there exist integers $a'_1, a'_2, a'_3$, not all zero, with $a'_1 r'_1 + a'_2 r'_2 + a'_3 r'_3 = 0$ and

$$|a'_1| + |a'_2| + |a'_3| < |a_1| + |a_2| + |a_3|.$$ 

After finitely many steps we must arrive at a case where one of the $a_i$ vanishes, in which case we finish as above.

If two of the $r_i$ are equal, then we are immediately done by choosing them as $x$ and $y$. Hence we may suppose $0 < r_1, r_2 < r_3$. Since we are free to negate all the $a_i$, we may assume $a_3 > 0$. Then either $a_1 < -\frac{1}{2} a_3$ or $a_2 < -\frac{1}{2} a_3$ (otherwise $a_1 r_1 + a_2 r_2 + a_3 r_3 > (a_1 + \frac{1}{2} a_3) r_1 + (a_2 + \frac{1}{2} a_3) r_2 > 0$). Without loss of generality, we may assume $a_1 < -\frac{1}{2} a_3$. Then choosing $x = r_1$ and $y = r_3$ gives the triple $(r'_1, r'_2, r'_3) = (r_1, r_2, r_3 - r_1)$ and $(a'_1, a'_2, a'_3) = (a_1 + a_3, a_2, a_3)$. Since $a_1 < a_1 + a_3 < \frac{1}{2} a_3 < -a_1$, we have $|a'_1| = |a_1 + a_3| < |a_1|$ and hence this operation has the desired effect. 

This problem was suggested by Kiran Kedlaya.

6. At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form $2^k$ for some positive integer $k$).

**Solution:** Let $n$ be the number of participants at the conference. We proceed by induction on $n$.

If $n = 1$, then we have one participant who can eat in either room; that gives us total of $2 = 2^1$ options.

Let $n \geq 2$. The case in which some participant, $P$, has no friends is trivial. In this case, $P$ can eat in either of the two rooms, so the total number of ways to split $n$ participants is
twice as many as the number of ways to split \((n - 1)\) participants besides the participant \(P\). By induction, the latter number is a power of two, \(2^k\); hence the number of ways to split \(n\) participants is \(2 \times 2^k = 2^{k+1}\), also a power of two. So we assume from here on that every participant has at least one friend.

We consider two different cases separately: the case when some participant has an odd number of friends, and the case when each participant has an even number of friends.

**Case 1:** *Some participant, \(Z\), has an odd number of friends.*

Remove \(Z\) from consideration and for each pair \((X, Y)\) of \(Z\)’s friends, reverse the relationship between \(X\) and \(Y\) (from friends to strangers or vice versa).

**Claim.** The number of possible seatings is unchanged after removing \(Z\) and reversing the relationship between \(X\) and \(Y\) in each pair \((X, Y)\) of \(Z\)’s friends.

**Proof of the claim.** Suppose we have an arrangement prior to \(Z\)’s departure. By assumption, \(Z\) has an odd number of friends in the room with him.

If this number is 0, the room composition is clearly still valid after \(Z\) leaves the room.

If this number is positive, let \(X\) be one of \(Z\)’s friends in the room with him. By assumption, person \(X\) also has an even number of friends in the same room. Remove \(Z\) from the room; then \(X\) will have an odd number of friends left in the room, and there will be an odd number of \(Z\)’s friends in this room besides \(X\). Reversing the relationship between \(X\) and each of \(Z\)’s friends in this room will therefore restore the parity to even.

The same reasoning applies to any of \(Z\)’s friends in the other dining room. Indeed, there will be an odd number of them in that room, hence each of them will reverse relationships with an even number of individuals in that room, preserving the parity of the number of friends present.

Moreover, a legitimate seating without \(Z\) arises from exactly one arrangement including \(Z\), because in the case under consideration, only one room contains an even number of \(Z\)’s friends.

Thus, we have to double the number of seatings for \((n - 1)\) participants which is, by the induction hypothesis, a power of 2. Consequently, for \(n\) participants we will get again a power of 2 for the number of different arrangements.

**Case 2:** *Each participant has an even number of friends.*
In this case, each valid split of participants in two rooms gives us an even number of friends in either room.

Let \((A, B)\) be any pair of friends. Remove this pair from consideration and for each pair \((C, D)\), where \(C\) is a friend of \(A\) and \(D\) is a friend of \(B\), change the relationship between \(C\) and \(D\) to the opposite; do the same if \(C\) is a friend of \(B\) and \(D\) is a friend of \(A\). Note that if \(C\) and \(D\) are friends of both \(A\) and \(B\), their relationship will be reversed twice, leaving it unchanged.

Consider now an arbitrary participant \(X\) different from \(A\) and \(B\) and choose one of the two dining rooms. [Note that in the case under consideration, the total number of participants is at least 3, so such a triplet \((A, B; X)\) can be chosen.] Let \(A\) have \(m\) friends in this room and let \(B\) have \(n\) friends in this room; both \(m\) and \(n\) are even. When the pair \((A, B)\) is removed, \(X\)’s relationship will be reversed with either \(n\), or \(m\), or \(m + n - 2k\) (for \(k\) the number of mutual friends of \(A\) and \(B\) in the chosen room), or 0 people within the chosen room (depending on whether he/she is a friend of only \(A\), only \(B\), both, or neither). Since \(m\) and \(n\) are both even, the parity of the number of \(X\)’s friends in that room will be therefore unchanged in any case.

Again, a legitimate seating without \(A\) and \(B\) will arise from exactly one arrangement that includes the pair \((A, B)\): just add each of \(A\) and \(B\) to the room with an odd number of the other’s friends, and then reverse all of the relationships between a friend of \(A\) and a friend of \(B\). In this way we create a one-to-one correspondence between all possible seatings before and after the \((A, B)\) removal.

Since the number of arrangements for \(n\) participants is twice as many as that for \((n - 2)\) participants, and that number for \((n - 2)\) participants is, by the induction hypothesis, a power of 2, we get in turn a power of 2 for the number of arrangements for \(n\) participants. The problem is completely solved.

This problem was suggested by Sam Vandervelde.