1. Prove that for each positive integer $n$, there are pairwise relatively prime integers $k_0, k_1, \ldots, k_n$, all strictly greater than 1, such that $k_0k_1 \cdots k_n - 1$ is the product of two consecutive integers.

2. Let $ABC$ be an acute, scalene triangle, and let $M$, $N$, and $P$ be the midpoints of $BC$, $CA$, and $AB$, respectively. Let the perpendicular bisectors of $AB$ and $AC$ intersect ray $AM$ in points $D$ and $E$ respectively, and let lines $BD$ and $CE$ intersect in point $F$, inside of triangle $ABC$. Prove that points $A$, $N$, $F$, and $P$ all lie on one circle.

3. Let $n$ be a positive integer. Denote by $S_n$ the set of points $(x, y)$ with integer coordinates such that

$$|x| + \left| y + \frac{1}{2} \right| < n.$$ 

A path is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_\ell, y_\ell)$ in $S_n$ such that, for $i = 2, \ldots, \ell$, the distance between $(x_i, y_i)$ and $(x_{i-1}, y_{i-1})$ is 1 (in other words, the points $(x_i, y_i)$ and $(x_{i-1}, y_{i-1})$ are neighbors in the lattice of points with integer coordinates).

Prove that the points in $S_n$ cannot be partitioned into fewer than $n$ paths (a partition of $S_n$ into $m$ paths is a set $\mathcal{P}$ of $m$ nonempty paths such that each point in $S_n$ appears in exactly one of the $m$ paths in $\mathcal{P}$).
4. Let $\mathcal{P}$ be a convex polygon with $n$ sides, $n \geq 3$. Any set of $n - 3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n - 2$ triangles. If $\mathcal{P}$ is regular and there is a triangulation of $\mathcal{P}$ consisting of only isosceles triangles, find all the possible values of $n$.

5. Three nonnegative real numbers $r_1, r_2, r_3$ are written on a blackboard. These numbers have the property that there exist integers $a_1, a_2, a_3$, not all zero, satisfying $a_1 r_1 + a_2 r_2 + a_3 r_3 = 0$. We are permitted to perform the following operation: find two numbers $x, y$ on the blackboard with $x \leq y$, then erase $y$ and write $y - x$ in its place. Prove that after a finite number of such operations, we can end up with at least one 0 on the blackboard.

6. At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form $2^k$ for some positive integer $k$).

Copyright © Committee on the American Mathematics Competitions, Mathematical Association of America