Math 115 Homework 9 Solutions

32.2 Let \( f(x) \),
\[
    f(x) = \begin{cases} 
        x & x \in \mathbb{Q} \\
        0 & x \notin \mathbb{Q}
    \end{cases}
\]

a) Calculate the upper and lower sums for \( f \) for the interval \([0, b]\).

Answer. It is easy to see that the lower sums are always zero. For the upper sum, first note that for any interval \([c, d]\),
\[
    M(f, [c, d]) = \sup\{f(x) | x \in [c, d]\} = d
\]

Thus, for any partition we have,
\[
    U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} t_k(t_k - t_{k-1})
\]

Now consider the function \( g(x) = \frac{x^2}{2} \) by the mean value theorem applied to the interval \([t_{k-1}, t_k]\), we have that
\[
    g(t_k) - g(t_{k-1}) = g'(z_k)(t_k - t_{k-1})
\]

for some \( z_k \in [t_{k-1}, t_k] \), and also \( g'(z_k) = z_k \) and it is clear that \( t_k \geq z_k \) thus
\[
    \sum_{k=1}^{n} t_k(t_k - t_{k-1}) \geq \sum_{k=1}^{n} z_k(t_k - t_{k-1})
\]

\[
    = \sum_{k=1}^{n} (g(t_k) - g(t_{k-1}))(t_k - t_{k-1}) = g(b) - g(0) = \frac{b^2}{2}
\]

Therefore, for any partition of \([0, b]\),
\[
    U(f, P) \geq \frac{b^2}{2}
\]

Thus \( U(f) \geq \frac{b^2}{2} \), by taking a equally spaced partition, \( t_k = \frac{kb}{n} \) we have that
\[
    U(f, P) = \sum_{k=1}^{n} t_k(t_k - t_{k-1}) = \sum_{k=1}^{n} \frac{kb}{n} \frac{1}{n} = \frac{b}{n^2} \sum_{k=1}^{n} k = \frac{b}{n^2} \frac{n(n+1)}{2}
\]

which tends to \( \frac{b^2}{2} \) thus
\[
    U(f) = \frac{b^2}{2}
\]

b) No.
32.6 Let \( f \) be a bounded function on \([a, b]\). Suppose there exists sequences \((U_n)\) and \((L_n)\) of upper and lower sums for \( f \) such that \( \lim_{n \to \infty} (U_n - L_n) = 0 \).

Show that \( f \) is integrable and \( \lim_{n \to \infty} U_n = \lim_{n \to \infty} L_n = \int_a^b f \).

Proof. We will use theorem 32.5. Let \( \epsilon > 0 \), since \( \lim_{n \to \infty} (U_n - L_n) = 0 \), we can find \( n \) such that \( U_n - L_n < \epsilon \).

Let's suppose that \( U_n = U(f, P) \) and that \( L_n = L(f, Q) \) for some partitions \( P, Q \). By the lemma 32.3 we have then,

\[
L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, P)
\]

and also,

\[
L(f, Q) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q)
\]

therefore,

\[
L_n = L(f, Q) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, P) = U_n
\]

since \( U_n - L_n < \epsilon \), then \( U(f, P \cup Q) - L(f, P \cup Q) < \epsilon \). Therefore by theorem 32.5, \( f \) is integrable.

Now for the second part of the exercise, first note that for every \( n, m \)

\[
U_n \geq U(f) = L(f) \geq L_m
\]

and then the limits \( \lim_{n \to \infty} U_n, \lim_{n \to \infty} L_n \) exists and are equal to the common value \( U(f) = L(f) = \int_a^b f \).

If the \( \lim_{n \to \infty} U_n \) is not \( U(f) \), then there exists a positive \( \epsilon \) such that there are infinitely many \( n \)'s such that \( U_n \) is bigger than \( U(f) + \epsilon \), but every \( L_n \) is less than \( U(f) \) so for infinitely many \( n \)'s we have

\[
L_n \leq U(f) < U(f) + \epsilon \leq U_n
\]

but this implies that for infinitely many \( n \)'s, \( U_n - L_n > \epsilon \), contradicting the fact that \( \lim_{n \to \infty} U_n - L_n = 0 \). □