Math 115 Homework 3 Solutions

7.4 a) Consider the sequence \( \chi_n = \frac{\sqrt{n}}{n} \), it is clear that this sequence tends to zero, which is a rational number, but every term is irrational.

8.4 Proof. First note that as \( |t_n| \leq M \), then \( 0 \leq M \), if \( M = 0 \) then \( (t_n) \) is a constant sequence with value zero, and it is clear that in this case \( \lim_{n \to \infty} (s_n t_n) = 0 \).

Now if \( M > 0 \), let \( \epsilon > 0 \), since \( \lim_{n \to \infty} s_n = 0 \), there exists \( N \) such that
\[
    n > N \implies |s_n| < \frac{\epsilon}{M} \tag{*}
\]

Now, for every \( n \) we have,
\[
|s_n t_n| = |s_n||t_n| < |s_n| M
\]

and if \( n > N \) then by (*)
\[
|s_n t_n| < \frac{\epsilon}{M} M = \epsilon
\]

\[\square\]

8.8b Note that,
\[
\sqrt{n^2 + n} - n = \frac{n}{\sqrt{n^2 + n} + n} = \frac{n}{n(\sqrt{1 + \frac{1}{n}} + 1)} = \frac{1}{(\sqrt{1 + \frac{1}{n}} + 1)}
\]

therefore,
\[
\left| (\sqrt{n^2 + n} - n) - \frac{1}{2} \right| = \left| \frac{1}{(\sqrt{1 + \frac{1}{n}} + 1)} - \frac{1}{2} \right| = \left| \frac{1}{2} - \sqrt{1 + \frac{1}{n}} \right|
\]

since,
\[
\left| \frac{1}{2} - \sqrt{1 + \frac{1}{n}} \right| \leq \left| \frac{1}{2} - \sqrt{1 + \frac{1}{n}} \right| = \left| \frac{(1 - \sqrt{1 + \frac{1}{n}})(1 + \sqrt{1 + \frac{1}{n}})}{2(1 + \sqrt{1 + \frac{1}{n}})} \right|
\]

\[
\leq \left| \frac{-1}{2n} \right| = \left| \frac{1}{2n} \right|
\]

\[
\leq \left| \frac{1}{2n} \right|
\]
therefore,

\[
\left| (\sqrt{n^2 + n} - n) - \frac{1}{2} \right| \leq \frac{1}{2n}
\]

thus we can take \( N = \frac{1}{2\epsilon} \) and we have that,

\[
n > N \text{ implies } \left| (\sqrt{n^2 + n} - n) - \frac{1}{2} \right| < \epsilon
\]

9.4b Suppose that \((s_n)\) converges, where \((s_n)\) satisfies

\[
s_{n+1} = \sqrt{s_n + 1}, \quad s_1 = 1
\]

(*)
as \( \lim_{n \to \infty} s_n \) exists, then \( \lim_{n \to \infty} s_{n+1} \) also exists, and in fact both limits are equal, call this value \( L \),

\[
\lim_{n \to \infty} s_n = \lim_{n \to \infty} s_{n+1} = L
\]

then from the relation (*), taking limits at both sides, we have that,

\[
L = \sqrt{L + 1}
\]

the solutions of this equation are,

\[
\frac{1 \pm \sqrt{5}}{2}
\]

we discard the negative one, because \( s_1 \) is non-negative and \( s_{n+1} \geq s_n \), which implies that the limit must also be non-negative.

9.10a Proof. As \( k > 0 \), then \( \frac{M}{k} > 0 \), since \( \lim_{n \to \infty} s_n = +\infty \), there exists \( N \) such that

\[
n > N \text{ implies } s_n > \frac{M}{k}
\]

therefore, for every \( n > N \)

\[
ks_n > k \frac{M}{k} = M
\]

which proves that \( \lim_{n \to \infty} ks_n = +\infty \) \( \square \)

9.18 a) Let \( S = 1 + a + a^2 + \ldots + a^n \), then

\[
aS = a + a^2 + \ldots + a^{n+1}
\]

therefore

\[
S - aS = 1 - a^{n+1}
\]

(*)&

but \( S - aS = (1-a)S \), as \( a \neq 1 \) we can divide (*) by \( 1-a \) and get

\[
S = \frac{1 - a^{n+1}}{1 - a}
\]
d) if \( a \geq 1 \), then
\[
\lim_{n \to \infty} (1 + a + a^2 + \ldots + a^n) = \infty
\]

because, if \( a = 1 \) this is the limit,
\[
\lim_{n \to \infty} (1 + 1 + \ldots + 1) = \lim_{n \to \infty} n = \infty
\]

if \( a > 1 \) we can use the identity found in a) and by the exercise 9.13
\[
\lim_{n \to \infty} a^n = \infty
\]

therefore
\[
\lim_{n \to \infty} (1 + a + a^2 + \ldots + a^n) = \lim_{n \to \infty} \frac{1 - a^{n+1}}{1 - a} = \infty
\]

10.6 \( a) \) for \( m > n \) we have that

\[
|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + s_{m-2} - \ldots + s_{n+1} - s_n|
\]

by the triangle inequality,

\[
|s_m - s_n| \leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \ldots + |s_{n+1} - S_n|
\]

we know that

\[
|s_{n+1} - s_n| < 2^{-n} \text{ for all } n
\]

therefore,

\[
|s_m - s_n| \leq \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \ldots + \frac{1}{2^n}
\]

taking out \( \frac{1}{2^n} \), we have,

\[
|s_m - s_n| \leq \frac{1}{2^n} \left( \frac{1}{2^{m-1-n}} + \frac{1}{2^{m-2-n}} + \ldots + \frac{1}{2^n} + 1 \right) \quad (*)
\]

but

\[
\left( \frac{1}{2^{m-1-n}} + \frac{1}{2^{m-2-n}} + \ldots + \frac{1}{2^n} + 1 \right) \leq 1 + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2^n} + \ldots
\]

and \( \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}} = 2 \) therefore from (\( \star \)),

\[
|s_m - s_n| \leq \frac{2}{2^n} = \frac{1}{2^{n-1}}
\]

Take \( N \) such that,

\[
n > N \implies \frac{1}{2^{n-1}} < \epsilon
\]
b) Consider the sequence

\[ s_n = \sum_{i=1}^{n} \frac{1}{i} \]

then

\[ |s_{n+1} - s_n| = \frac{1}{n+1} < \frac{1}{n} \]

but this sequence does not converges, it diverges to infinity (integral test, for example). So it cannot be a Cauchy sequence, because in \( \mathbb{R} \) a cauchy sequence is always converges to a real number.

12.8 Let \( \alpha_n = \sup\{s_m \mid m > n\} \), and \( \beta_n = \sup\{t_m \mid m > n\} \), then for every \( m > n \),

\[ s_m + t_m \leq \alpha_n + \beta_n \]

therefore by the definition of supremum,

\[ \sup\{s_m + t_m \mid m > n\} \leq \alpha_n + \beta_n \]

taking limits at both sides and recalling that by definition,

\[ \lim sup s_n = \lim_{n \to \infty} \sup\{s_n \mid n > N\} \]

we have

\[ \lim_{n \to \infty} \sup\{s_m + t_m \mid m > n\} \leq \lim_{n \to \infty} (\alpha_n + \beta_n) = \lim_{n \to \infty} \alpha_n + \lim_{n \to \infty} \beta_n \]

therefore

\[ \lim sup(s_n + t_n) \leq \lim sup s_n + \lim sup t_n \]