34.2 (a) If we let $F(x) = \int_0^x e^{t^2} \, dt$, then we see that
\[
\lim_{x \to 0} \int_0^x e^{t^2} \, dt = \lim_{x \to 0} \frac{F(x) - F(0)}{x - 0}.
\]
So the limit in question is the derivative of $F(x)$ at $x = 0$. By the fundamental theorem of calculus 2, it is $e^0 = 1$. Same story for part (b). The answer in this case is $e^9$.

34.5 We’re going to use the fact that $\int_a^b f(t) \, dt = -\int_b^a f(t) \, dt$. Let $G(x) = \int_a^x f(t) \, dt$. Then $G'(x) = f(x)$. Using the aforementioned fact we see that $F(x) = G(x + 1) - G(x - 1)$ and so $F'(x) = G'(x + 1) - G'(x - 1) = f(x + 1) - f(x - 1)$.

34.6 Let $F(x) = \int_0^x f(t) \, dt$. Then $F(\sin(x)) = G(x)$, and hence $G'(x) = F'(\sin(x)) \cos(x)$ be the chain rule. Then by the fundamental theorem of calculus 2, we see $G'(x) = f(\sin(x)) \cos(x)$.

34.11 I talked about this in class. Here’s the proof I had in mind. If I could find a way to draw nice importable pictures easily I would, but here goes the purely algebraic proof. We prove the contrapositive. Suppose that $f(x_0) > 0$ for some $x_0$. We will show that $\int_a^b f > 0$ as well. Let $\epsilon = f(x_0)/2$. Since $f$ is continuous at $x_0$, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. It follows that for all $x \in [x_0 - \delta, x_0 + \delta]$, $f(x) > 0$, so that $m(f, [x_0 - \delta, x_0 + \delta]) = \inf\{f(x) : x \in [x_0 - \delta, x_0 + \delta]\} > 0$. Choose a partition $P$ such that $P = \{a < t_1 < \cdots < x_0 - \delta < x_0 + \delta < \cdots < b\}$. Then clearly $L(f, P) > 0$ since $f \geq 0$ and $m(f, [x_0 - \delta, x_0 + \delta]) \geq 0$. Since $\int_a^b f = L(f) = \sup\{L(f, P) : P \text{ a partition}\}$ and $L(f, P) > 0$ for the $P$ chosen above, this supremum must also be positive, and hence the integral is positive.

34.12 We’re going to use the previous exercise here. We are supposing that $f$ is a continuous function on $[a, b]$ and that for every continuous function $g$ on $[a, b]$, $\int_a^b f(t)g(t) \, dt = 0$. In particular this is true if $g(x) = f(x)$, and hence $\int_a^b (f(t))^2 \, dt = 0$. Since $f$ is continuous, so is $f^2$, and by exercise 34.11, $(f(x))^2 = 0$ for every $x$, and hence $f(x) = 0$ for every $x$. 

Math 115, Fall 2003
Homework 11
Selected Solutions