I. Show that a smooth $n$-manifold can be embedded as a submanifold and a closed subset of $\mathbb{R}^{2n+1}$.

(Hint: Let $h$ be a smooth proper map $h : M \to \mathbb{R}$. Use the method we discussed to prove Whitney embedding theorem to find a proper embedding $f : M \to \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R}$.)

II. Let $M$ be a smooth $n$-manifold (i.e. $M$ is a topological manifold equipped with a particular smooth structure $[A]$ which we omit from the notation.)

Definition A: An A-orientation of $M$ is an orientation $\mu$ of the underlying topological manifold. More precisely, $\mu$ is a family of generators $\mu_x \in H_n(M, M - \{x\})$. They are required to satisfy a continuity condition, cf. Hatcher.

Definition B: An atlas $A$ in the smooth structure of $M$ is oriented if $\det(D(h'oh^{-1})(x)) > 0$ for all $h, h' \in A$ and all $x$ for which $h'oh^{-1}(x)$ is defined. Two oriented atlases $A, A_0$ are equivalent if the union $A \cup A_0$ is again an oriented atlas. A B-orientation on $M$ is an equivalence class of oriented atlases.

Definition C: A C-orientation on $M$ is a form $\omega(M) \in \Omega^n(M)$ such that $\omega(x) \neq 0 \in \text{Alt}^n(T_xM)$ for all $x \in M$.

Given an A-orientation of $M$, we let $A_{\text{max}}$ be the maximal atlas for the smooth structure. Let $A(\mu) = \{(h, U, U') \mid h : U \to U' \text{ preserves local atlas}\}$.

Prove that $\mu \to A$ gives a bijection between the set of A-orientations of $M$ and the set of B-orientations. Also, construct a bijection between the set of B-orientations and the set of C-orientations on $M$.

III. Let $X \subset \mathbb{R}^m$ be a locally closed set, $N \subset \mathbb{R}^n$ a boundaryless smooth manifold and $f : X \to N$ a continuous proper mapping. Show that there is a positive continuous function $\epsilon : X \to \mathbb{R}$ such that every continuous mapping $g : X \to N$ with $|f(x) - g(x)| < \epsilon(x)$ for all $x \in X$ is properly homotopic to $f$.

Remark. A proper homotopy is simply a homotopy $H : [0, 1] \times M \to N$ that is a proper mapping.

IV. Let $M$ and $N$ be smooth manifolds and $g : M \to N$ be a continuous function. Let $A$ be a closed subset of $M$. Show that $g$ is homotopic relative to $A$ to a map that is smooth on $M - A$.

V. Suppose $M$ is a smooth, compact manifold that admits a nowhere vanishing vector field. Show that there exists a smooth map $F : M \to M$ that is homotopic to the identity and has no fixed points.
6. Let $C$ be a compact subset of $[0, 1]$ with measure zero. Show $C$ is equal to the set of critical values of a $C^1$ function $g : \mathbb{R} \to \mathbb{R}$. Is the same statement true if we replace $C^1$ by $C^\infty$?

7. Construct a $C^1$ map $f : \mathbb{R}^2 \to \mathbb{R}$ such that the set of critical values of $f$ contains an open interval $I \subset \mathbb{R}$.

*Hint:* Let $B = \{ \sum_i a_i \cdot \frac{3^i}{2^i} \mid a_i \in \{0, 2\} \} \subset [0, 1]$ be the Cantor ternary set. Construct a differentiable function $h : \mathbb{R} \to \mathbb{R}$ such that for any $y \in B$, $h'(y) = 0$ and $h(y) \in B$, and $h \in C^1$. 

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