Problem 1- Exam day 1: Make the change of variables
\[
\begin{align*}
  u &= x + y \\
  v &= x - y
\end{align*}
\]
Then the new region would be
\[
\begin{align*}
  0 &\leq u \leq 5 \\
  0 &\leq v \leq 3
\end{align*}
\]
and our Jacobian is
\[
\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{2}
\]
So our integral becomes
\[
\frac{1}{2} \int_0^3 \int_0^5 u e^{uv} \, du \, dv = \frac{1}{2} \int_0^5 (e^{uv}) \bigg|_0^3 \, du = \frac{1}{2} \int_0^5 (e^{3u} - 1) \, du = \frac{e^{15} - 16}{6}
\]
Problem 2- Exam day 1: Let \( F = Pi + Qj + Rk \) where
\[
\begin{align*}
  P &= x^2 + y^2 + z^2 - 6x, \\
  Q &= \frac{y}{x^2 + y^2 + z^2}, \\
  R &= \frac{z}{x^2 + y^2 + z^2}. 
\end{align*}
\]
To show that \( F \) is conservative it is enough to consider the function
\[
f(x,y,z) = \frac{1}{2} \ln(x^2 + y^2 + z^2) - 3x^2.
\]
Since
\[
\begin{align*}
  f_x &= P, \\
  f_y &= Q, \\
  f_z &= R,
\end{align*}
\]
\( F \) is conservative. Then by fundamental theorem for line integrals we get
\[
I = \int_C F \cdot dr = f(r(1)) - f(r(0)) = f(0,0,1) - f(1,0,0) = 3
\]
Problem 3-Exam day 1: By Divergence Theorem, we have
\[
\int \int_S F \cdot dS = \int \int \int_E (3x^2 + 3y^2 + 3) \, dV.
\]
The region \( E \) can be described by \( z \geq 0 \) and \( x^2 + y^2 + z^2 \leq 4 \). In terms of the spherical coordinates \( E \) is given by
\[
1 \leq \rho \leq 2, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi/2.
\]
By converting the triple integral to spherical coordinates, we get
\[
\int \int \int_E (3x^2 + 3y^2 + 3) \, dV = 6\pi \int_1^2 \int_0^{\pi/2} \int_0^{2\pi} (\rho^2 \sin^2(\phi) + 3) \rho^2 \sin(\phi) \, d\theta \, d\phi \, d\rho = (194/5)\pi
\]
Problem 3- Exam day 2:
We can write the limit of variables as
\[
0 \leq y \leq 3, \quad 0 \leq x \leq \sqrt{9 - y^2}, \quad \sqrt{x^2 + y^2} \leq z \leq \sqrt{18 - x^2 - y^2}.
\]
In spherical coordinates, the integral equals
\[
\int_{0}^{\pi/4} \int_{0}^{\pi/2} \int_{0}^{\sqrt{18}} \rho^3 \sin(\phi) d\rho d\theta d\phi.
\]

**Problem 4 - Exam day 1:** Be VERY careful with orientation when applying Green’s theorem. In this problem, the semicircle is oriented clockwise, whereas Green’s theorem applies to curves oriented counterclockwise.

Note that the boundary of the upper half disk consists of a bottom line and the semicircular arc in question. Denote by \(-C\) the arc oriented counterclockwise, and let \(L\) be the bottom line. Also, let \(D^+\) denote the upper half disk.

Then, letting \(P(x, y) = y(2x^2 + 1)e^{x^2}\), \(Q(x, y) = x(1 + e^{x^2})\), Green’s theorem tells us that

\[
\int_C Pdx + Qdy + \int_L Pdx + Qdy = \int \int_{D^+} (Q_x - P_y) dy dx
\]

By some computations we find that \(Q_x - P_y = 1\), and that \(\int_L Pdx + Qdy = 0\).

Thus

\[
\int_C Pdx + Qdy = -\int_{-C} Pdx + Qdy = -\int_{D^+} 1 dy dx = -\text{area of } D^+ = -2\pi
\]

**Problem 5 - Exam day 1:** Since \(S\) can be parametrized by \(r(x, y) = (x, y, g(x, y))\) we have \(N = (-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1)\). Then,

\[
\int_S F \cdot dS = \int_D F \cdot N dA = \int_D (0, 0, 1) \cdot (-g_x, -g_y, 1) dA = \int_D 1 dA = \text{Area}(D)
\]

**Problem 5 - Exam day 2:**

\[
\mathbf{r}_u \times \mathbf{r}_v = \sin(v)\mathbf{i} - \cos(v)\mathbf{j} + u\mathbf{k},
\]

therefore

\[
\mathbf{F}(r(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = u^2 v(\cos(v) + \sin(v)).
\]

Then we have

\[
\int \int_S F dS = \int_0^\pi \int_0^1 u^2 v(\cos(v) + \sin(v)) du dv =
\]

\[
= \frac{1}{3} \int_0^\pi v(\cos(v) + \sin(v)) dv = \frac{1}{3} (v(\sin(v) - \cos(v)))|_0^\pi = \pi/3 - 2/3
\]

**Problem 6** The intersection curve \(\gamma\) of the ellipsoid and the plane is given by the equation

\[
(x - 1)^2 + y^2 + 1, \hspace{1cm} z = 2 - x.
\]
Let $\Sigma$ be the portion of the plane $z = 2 - x$ that lies within $\gamma$, with the upward orientation. Then, by Stokes theorem,

$$I = \int_C (yz\mathbf{i} + 2xz\mathbf{j} + 3xy\mathbf{k}) \cdot d\mathbf{r} = \int \int_{\Sigma} \nabla \times (z\mathbf{i} + 2xz\mathbf{j} + 3xy\mathbf{k}) \cdot d\mathbf{S}.$$ 

On the other hand,

$$\nabla \times (z\mathbf{i} + 2xz\mathbf{j} + 3xy\mathbf{k}) = xi - 2yj + zk.$$

Also $\Sigma$ is the graph of $z = 2 - x$ for $(x, y)$ in the disk with equation $(x - 1)^2 + y^2 \leq 1$. Therefore, we have

$$I = \int \int_D (x + 0 + (2 - x)) dA = 2 \times \text{Area}(D) = 2\pi.$$

**Problem 7- Exam day 1:**

The surface $S$ is defined by $ax^2 + by^2 + cz^2 - 1 = 0$, so the unit normal vector can be written as

$$\mathbf{n} = \frac{ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}}{\sqrt{a^2 x^2 + b^2 y^2 + c^2 z^2}}.$$

Let

$$\mathbf{F} = \frac{x}{a}\mathbf{i} + \frac{y}{b}\mathbf{j}.$$

Then

$$\mathbf{F} \cdot \mathbf{n} = \frac{x^2 + y^2}{\sqrt{a^2 x^2 + b^2 y^2 + c^2 z^2}}.$$

Let

$$V = \{(x, y, z) \mid ax^2 + by^2 + cz^2 \leq 1\}.$$

Note that $\text{div}(\mathbf{F}) = \frac{1}{a} + \frac{1}{b}$, Now by using Divergence theorem, we get

$$I = \int \int_S \frac{x^2 + y^2}{\sqrt{a^2 x^2 + b^2 y^2 + c^2 z^2}} dS = \int \int \int_V \text{div}(\mathbf{F}) dV = \left(\frac{1}{a} + \frac{1}{b}\right) \text{Vol}(V).$$

On the other hand, $\text{Vol}(V) = \frac{4}{3 \sqrt{abc}}$ and hence

$$I = \frac{1}{a} + \frac{1}{b} \left(\frac{4}{3 \sqrt{abc}}\pi\right).$$

(In order to calculate $\text{Vol}(V)$, you can use the change of variables $u = \frac{x}{\sqrt{a}}$, $v = \frac{y}{\sqrt{b}}$, $w = \frac{z}{\sqrt{c}}$ and use the formula for the volume of sphere.)

**Problem 8:** The plane $z = -1$ intersects the given sphere at $x^2 + y^2 = 3$. Let $E = \{(x^2 + y^2 \leq 3, z = -1)\}$, and let $V$ be the solid bounded by $S$ and $E$, then divergence theorem tells us that

$$\int \int_S \mathbf{F} \cdot d\mathbf{S} + \int \int_E \mathbf{F} \cdot d\mathbf{S} = \int \int \int_V \text{div} \mathbf{F} dV = \int \int \int_V 1 + 2 - 3 dV = 0$$

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so we have
\[ \int \int_S F \cdot \mathbf{n} \, dS = - \int \int_E F \cdot \mathbf{n} \, dS \]

Now on \( E \), the outward normal is \(-\mathbf{k}\) and \( z = -1 \). Therefore
\[ \int \int_E F \cdot \mathbf{n} \, dS = \int \int_{\{x^2 + y^2 \leq 3\}} -(2x^3 + y^3 - 3(-1)) \, dxdy \]

By symmetry, the integration of \( x^3 \) and \( y^3 \) vanishes, and we are left with
\[ \int \int_{\{x^2 + y^2 \leq 3\}} (-3) \, dydx = (-3)(3\pi) = -9\pi \]

so the outward flux of \( F \) through \( S \) is given by
\[ \int \int_S F \cdot \mathbf{n} \, dS = - \int \int_E F \cdot \mathbf{n} \, dS = 9\pi \]

**Problem 9**: Observe that \( 1 - x^2 < 1 - y^2 \) if and only if \( |y| < |x| \). By symmetry the volume of \( T \) is equal to four times the volume of the region lying between the lines \( x = y \) and \( x = -y \) in the first and fourth quadrants. Since \( |x| > |y| \) in the given region we get,
\[
\text{Vol}(T) = 4 \int_0^1 \int_{-x}^x \int_0^{1-x^2} 1 \, dz dx dy = 4 \int_0^1 \int_{-x}^x (1 - x^2) dy dx = 4 \int_0^1 2x - 2x^3 dx = 4 \left[ x^2 - \frac{1}{2} x^4 \right]_0^1 = 2
\]

Then by divergence theorem
\[
\Phi = \int \int_S F \cdot dS = \int \int_T \nabla F \cdot dV = \int \int_T (2xz + 1 + 3 - 2xz) dV = \int \int_T 4 \, dV = 4 \text{Vol}(T) = 8
\]

**Problem 10** By Stokes' theorem, if \( S_1 \) is the portion of \( S \) enclosed by \( C \), then
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_{S_1} \text{Curl}(\mathbf{F}) \cdot \mathbf{n} \, dS.
\]

On the other hand,
\[
\text{Curl}(\mathbf{F}) = -z\mathbf{j}
\]

and \( \mathbf{N} = -f'(x)i + 0j + k \). Therefore, on \( S_1 \), we have \( \text{Curl}(\mathbf{F}) \cdot \mathbf{n} = 0 \). So
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_{S_1} 0 \, dS = 0.
\]