Solutions to Homework 2
MATH52

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First express the boundaries in terms of cylindrical coordinates:
\[
\begin{aligned}
\{ r^2 &\leq z \\
2r \cos \theta &\geq z \\
\end{aligned}
\]

Combining these inequalities gives us the region of integration:

\[
r^2 \geq 2r \cos \theta \iff r \leq 2 \cos \theta; \cos \theta \geq 0
\]

Thus we should consider an integral of the form
\[
\int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \int_0^{r^2} r^2 dz \, r \, dr \, d\theta
\]

\[
= 4 \int_{-\pi/2}^{\pi/2} \cos^3 \theta \cos \theta \, d\theta = 4 \int_{-\pi/2}^{\pi/2} \frac{1}{4} \cos 3 \theta + \frac{3}{4} \cos \theta \cos \theta \, d\theta
\]

\[
= \frac{1}{3} \int_{-\pi/2}^{\pi/2} \cos 3 \theta \cos \theta \, d\theta + \int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta
\]

\[
= \frac{1}{3} \int_{-\pi/2}^{\pi/2} \frac{1}{2} (\cos 4 \theta + \cos 2 \theta) \, d\theta + \int_{-\pi/2}^{\pi/2} \frac{1}{2} (1 + \cos 2 \theta) \, d\theta
\]

\[
= \frac{1}{6} \left( \frac{\sin 4 \theta}{4} + \frac{\sin 2 \theta}{2} \right) \bigg|_{-\pi/2}^{\pi/2} + \frac{1}{2} \left( \theta + \frac{\sin 2 \theta}{2} \right) \bigg|_{-\pi/2}^{\pi/2} = \frac{\pi}{2}
\]

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Set up the cylinder $C$ in coordinates as follows (assuming its height to be $h$):

\[
C = \{ (r, \theta, z) \mid r \leq a, \ 0 \leq \theta \leq 2\pi, \ 0 \leq z \leq h \}
\]

Suppose the cylinder has uniform density $\rho$ and recall that the moment of inertia is just the integral of $r^2 \rho$ over the cylinder, then the answer is given by:

\[
\int_0^{2\pi} \int_0^a \int_0^h r^2 rdzdrd\theta = (\text{the } \theta \text{- and } h \text{- integrations can be done first}) \ 2\pi h \int_0^a r^3 \rho dr
\]

\[
= \frac{\pi a^4 \rho h}{2}
\]
Note that the mass, \( m \), of the cylinder is \( \rho \) times its volume \( \pi a^2 h \). Substituting this into the expression for the moment of inertia, we get

\[
\frac{ma^2}{2},
\]
which is the desired result.

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Since the density in spherical coordinates is \( \delta = kr \cos \phi \), we know that the mass is given by

\[
m = \int_0^a \int_0^{2\pi} \int_0^{\pi/2} k r \cos \phi r^2 \sin \phi d\phi d\theta dr = k(2\pi)(\frac{a^4}{4}) \int_0^{\pi/2} \cos \phi \sin \phi d\phi
\]

\[
= \frac{k \pi a^4}{2} \left( \frac{1}{4} (-\cos 2\phi) \right|_{0}^{\pi/2} = \frac{k \pi a^4}{4}
\]

Notice that the mass distribution of the hemisphere is symmetric with respect to the \( z \)-axis, so the centroid shouldn’t change under rotation about that axis. This means that the centroid must be of the form

\((0, 0, z_0)\)

where \( z_0 = \frac{1}{m} \int_0^a \int_0^{2\pi} \int_0^{\pi/2} k(r \cos \phi)^2 r^2 \sin \phi d\phi d\theta dr \)

Evaluating the triple integral gives

\[
2k \pi a^5 \left( \frac{2}{5} \cos^2 \phi \sin \phi d\phi = \frac{2k \pi a^5}{5} \left( \frac{-\cos^3 \phi}{3} \right|_{0}^{\pi/2} = \frac{2k \pi a^5}{15}
\]

Thus \( z_0 = \frac{2k \pi a^5}{15} = \frac{8a}{15} \)

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We set up the spherical shell in terms of spherical coordinates as follows

\[
a \leq \rho \leq 2a
0 \leq \theta \leq 2\pi
0 \leq \phi \leq \pi
\]

Since its moment of inertia around each diameter is the same, we calculate it using the \( z \)-axis. Notice that the distance to the axis of the point \((\rho, \theta, \phi)\) is \( \rho \sin \phi \)

\[
I = \int_a^{2a} \int_0^{2\pi} \int_0^{\pi} (\rho \sin \phi)^2 (\rho^2) (\rho^2 \sin \phi) d\phi d\theta d\rho
\]
\[
\int_a^{2a} \int_0^{2\pi} \int_0^\pi \rho^6 \sin^3 \phi d\phi d\theta d\rho = 2\pi \frac{127a^7}{7} \int_0^\pi (1 - \cos^2 \phi) \sin \phi d\phi \\
= \frac{254\pi a^7}{7} \left( -\cos \phi + \frac{\cos^3 \phi}{3} \right) \bigg|_0^\pi = \frac{1016\pi a^7}{21}
\]

On the other hand the mass of the shell is
\[
m = \int_a^{2a} \int_0^{2\pi} \int_0^\pi (\rho^2)(\rho^2 \sin \phi) d\phi d\theta d\rho = 2\pi \frac{31a^5}{5} (2) = \frac{124\pi a^5}{5}
\]

So in terms of \( m \), the moment of inertia is \( \frac{1270a^2}{651} \)

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The cone is defined by
\[
0 \leq \theta \leq 2\pi \\
0 \leq \phi \leq \frac{\pi}{6} \\
0 \leq \rho \leq 2\cos \phi
\]

So we calculate its moment of inertia around z-axis by
\[
\int_0^{2\pi} \int_0^{\pi/6} \int_0^{2a \cos \phi} (\rho \sin \phi)^2 (\rho \cos \phi) (\rho^2 \sin \phi) d\rho d\phi d\theta = 2\pi \int_0^{\pi/6} \frac{1}{6} (2\cos \phi)^6 \sin^3 \phi \cos \phi d\phi \\
= \frac{64\pi a^6}{3} \int_0^{\pi/6} \cos^6 \phi (1 - \cos^2 \phi) \sin \phi \cos \phi d\phi \\
= \frac{64\pi a^6}{3} \int_0^{\pi/6} (\cos^7 \phi - \cos^9 \phi) \sin \phi d\phi = \frac{64\pi a^6}{3} \left( -\frac{\cos^8 \phi}{8} + \frac{\cos^{10} \phi}{10} \right) \bigg|_0^{\pi/6} = \frac{47\pi a^6}{240}
\]

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We’ll use cylindrical coordinates \((r, \theta, z)\). From figure 13.7.15 (b) we see that the solid in question is made up by eight pieces, each congruent to the portion lying in \( 0 \leq \theta \leq \pi/4 \). Thus, we need only compute the volume of this particular portion, which we denote by \( R \).

The precise description of \( R \) in terms of cylindrical coordinates is as follows
\[
0 \leq \theta \leq 4\pi \\
0 \leq \rho \leq 1 \\
-\sqrt{1 - \rho^2 \cos^2 \theta} \leq z \leq \sqrt{1 - \rho^2 \cos^2 \theta}
\]
and the volume of $R$ is equal to

$$
\int_0^1 \int_0^{\pi/4} \int_{\sqrt{1-\rho^2\cos^2\theta}}^\sqrt{1-\rho^2\cos^2\theta} \rho 
\rho dz \rho d\theta \rho d\rho = 2 \int_0^1 \int_0^{\pi/4} \sqrt{1-\rho^2\cos^2\theta} \rho d\theta \rho d\rho
$$

we integrate $\rho$ first, making the following change of variables

$$
u = 1 - \rho^2 \cos^2 \theta 
\frac{du}{d\rho} = -2 \rho \cos^2 \theta d\rho
$$

We get

$$
2 \int_0^{\pi/4} \int_1^{1-\cos^2 \theta} \sqrt{u} \frac{1}{-2 \cos^2 \theta} dud\theta = 2 \int_0^{\pi/4} \int_{\sin^2 \theta}^{1} \sqrt{u} \frac{1}{2 \cos^2 \theta} dud\theta
$$

$$
2 \int_0^{\pi/4} \frac{1}{3 \cos^2 \theta} d\theta = 2 \int_0^{\pi/4} \frac{1}{\sec^2 \theta} d\theta - 2 \int_0^{\pi/4} \frac{\sin \theta (1 - \cos^2 \theta)}{\cos^2 \theta} d\theta
$$

$$
= \frac{2}{3} \tan \theta \Bigg|_0^{\pi/4} - \frac{2}{3} \int_0^{\pi/4} (\tan \theta \sec \theta - \sin \theta) d\theta
$$

$$
= \frac{2}{3} - \frac{2}{3} (\sec \theta + \cos \theta) \Bigg|_0^{\pi/4} = \frac{2}{3} [1 - ((\sqrt{2} - 1) + \frac{1}{\sqrt{2}} - 1)] = \frac{2}{3} (3 - \frac{3\sqrt{2}}{2})
$$

So the volume of the whole solid would be

$$
\frac{16}{3} (3 - \frac{3\sqrt{2}}{2}) = 8 (2 - \sqrt{2})
$$

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The region on the $(x, y)$-plane is given by

$$
0 \leq x \leq 1 
\sqrt{x} \leq y \leq \sqrt{x}
$$

Thus the area in question is given by

$$
\int_0^1 \int_{\sqrt{x}}^{x^2} \sqrt{1 + 2^2 + 2^2} dy dx = 3 \int_0^1 (\sqrt{x} - x^2) dx = 3 \left( \frac{2}{3} - \frac{1}{3} \right) = 1
$$

13.8 8

The problem just asks us to compute the area of the plane $z = 6 - 2x - 3y$ that lies above $x^2 + y^2 \leq 2$. The answer is given by the following integral
\[
\int \int_{x^2+y^2 \leq 2} \sqrt{1+2^2+3^2} dxdy = \sqrt{14} \text{ (area of the disk } \{x^2+y^2 \leq 2\}) = \sqrt{14}\sqrt{2\pi} = 2\sqrt{14}\pi
\]

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\[
\int \int_{x^2+y^2 \leq 4} \sqrt{1+(2x)^2+(-2y)^2} dxdy = (\text{use polar coordinates}) \int_0^2 \int_0^{2\pi} \sqrt{1+4r^2} r dr d\theta \\
= 2\pi \left( 1 + \frac{12}{3} \right) = \frac{8}{3}(1 + 4r^2)^{3/2} \bigg|_0^2 = \frac{\pi}{6} (17\sqrt{17} - 1)
\]

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We’ll do this using cylindrical coordinates. Recall that the area element in cylindrical coordinates is

\[dA = \sqrt{r^2 + r^2(\partial_c z)^2 + (\partial_\theta z)^2}\]

Also, notice that the area of the sphere cut out by the cylinder has two separate components, one in \(\{z > 0\}\) and the other in \(\{z < 0\}\). We need only calculate the first part and multiply by 2. Thus the area in question is

\[
2 \int_0^\pi \int_0^{a \sin \theta} \sqrt{r^2 + r^2 \left( \frac{r^2}{(a^2 - r^2)} \right)} dr d\theta = 4 \int_0^{\pi/2} \int_0^{a \sin \theta} \frac{ar}{\sqrt{a^2 - r^2}} dr d\theta \\
= 4a \int_0^{\pi/2} \left( -\sqrt{a^2 - r^2} \right)_{0}^{a \sin \theta} d\theta = 4a \int_0^{\pi/2} (a - a \cos \theta) d\theta \\
= 4a^2 (\frac{\pi}{2} - 1) = 2a^2 (\pi - 2)
\]

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Let \(X = (a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi)\), then we have

\[
X_\phi = (a \cos \phi \cos \theta, a \cos \phi \sin \theta, -a \sin \phi) \\
X_\theta = (-a \sin \phi \sin \theta, a \sin \phi \cos \theta, 0)
\]

therefore their cross product is

\[
X_\phi \times X_\theta = (a^2 \sin^2 \phi \cos \theta, a^2 \sin^2 \phi \sin \theta, a^2 \cos \phi \sin \phi)
\]

and the area element is

\[|X_\phi \times X_\theta| = a^2 \sin \phi\]
Hence the area over the \((\theta, \phi)\)-region is given by

\[
\int \int_R a^2 \sin \phi d\phi d\theta
\]

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(a) The \((\theta, \phi)\)-region corresponding to the whole sphere is

\[
0 \leq \theta \leq 2\pi \\
0 \leq \phi \leq \pi
\]

Thus the surface area of the sphere is

\[
\int_0^{2\pi} \int_0^{\pi} a^2 \sin \phi d\phi d\theta = 2\pi a^2 (-\cos \phi) \bigg|_0^\pi = 4\pi a^2
\]

(b)

\[
\int_0^{2\pi} \int_0^{\pi/6} a^2 \sin \phi d\phi d\theta = 2\pi a^2 (-\cos \phi) \bigg|_0^{\pi/6} = \pi a^2 (2 - \sqrt{3})
\]