1 Introduction

In this paper we investigate the dynamics of the earthquake flow defined by Thurston on the bundle $\mathcal{PM}_g$ of geodesic measured laminations. This flow is a natural generalization of twisting along simple closed geodesics. We discuss the relationship between the Teichmüller horocycle flow on the bundle $\mathcal{QM}_g$ of holomorphic quadratic differentials, and the earthquake flow. In fact, the basic ergodic properties of the Teichmüller horocycle flow are better understood [17]. In this paper, we show:

**Theorem 1.1.** The earthquake flow and the Teichmüller horocycle flow are measurably isomorphic. □

This result relies heavily on the work of Thurston [35] and Bonahon [2].

1.1 Notation

Let $S_g$ be a surface of genus $g$, and $\mathcal{T}_g$ be the Teichmüller space, the space of hyperbolic Riemann surfaces marked by $S_g$. It is known that the Teichmüller space is a complex manifold and the cotangent space at $X \in \mathcal{T}_g$ is the vector space $\mathcal{Q}(X)$ of holomorphic quadratic differentials at $X$. The Weil–Petersson symplectic form induces a measure $\mu_{wp}$ on $\mathcal{T}_g$. 
Let $\mathcal{ML}_g$ and $\mathcal{MF}_g$ respectively denote the space of measured laminations, and the space of measured foliations on $S_g$ \cite{8}. The space $\mathcal{ML}_g$ carries a mapping class group invariant volume form $\mu_{Th}$. Given $X \in \mathcal{T}_g$, and $\lambda \in \mathcal{ML}_g$, we denote the length of the lamination $\lambda$ in the hyperbolic metric on $X$ by $\ell_{\lambda}(X)$. Attached to $S_g$ one has:

- $Q T_g \rightarrow \mathcal{T}_g$, the bundle of holomorphic quadratic differentials;
- $Q^1 \mathcal{T}_g$, the unit sub-bundle for the norm
  \[ \|q\| = \int_X |q|; \]
- $Q^1 \mathcal{M}_g = Q^1 \mathcal{T}_g / \text{Mod}_g$, this space carries a natural finite measure $\mu_g$;
- $\mathcal{PT}_g = \mathcal{ML}_g \times \mathcal{T}_g$, the bundle of geodesic measured laminations over $\mathcal{T}_g$;
- $\mathcal{P}^1 \mathcal{T}_g$, the unit sub-bundle for the norm
  \[ \| (\lambda, X) \| = \ell_{\lambda}(X); \]

and finally,
- $\mathcal{P}^1 \mathcal{M}_g = \mathcal{P}^1 \mathcal{T}_g / \text{Mod}_g$ which carries a natural finite measure denoted by $\nu_g$.

In this paper, we will discuss the relationship between natural mapping class group invariant measures on $\mathcal{ML}_g$, $\mathcal{T}_g$, $Q^1 \mathcal{T}_g$, and $\mathcal{P}^1 \mathcal{T}_g$.

We wish to compare the dynamics of the earthquake flow on $\mathcal{P}^1 \mathcal{M}_g$ to the Teichmüller horocycle flow on $Q^1 \mathcal{M}_g$. These flows are defined as follows.

- Thurston’s earthquake flow on $\mathcal{PT}_g$ is defined at time $t$ by
  \[ \text{tw}^t(\lambda, X) = (\lambda, \text{tw}^t_{\lambda}(X)), \]
  where for a simple closed geodesic $\gamma$ on $X \in \mathcal{T}_g$, $\text{tw}^t_{\gamma}(X) \in \mathcal{T}_g$ is constructed by cutting $X$ along $\gamma$, twisting distance $t$ to the right, and re-gluing \cite{38}. Thurston has defined the general earthquake associated with a general measured geodesic lamination instead of the simple closed curve $\gamma$ \cite{14, 34}. As we will see later, the measure $\nu_g$ is invariant under the earthquake flow \S 3.
- Teichmüller horocycle flow $h^t$ is defined on $Q T_g$ as follows. Note that any holomorphic quadratic differential $q \in Q T_g$ can be defined with a pair of transverse measured foliations on $S_g$ : $\text{Re}(q)$ the horizontal foliation, and $\text{Im}(q)$ the vertical foliations of $q$ \cite{11}. For notational brevity, we denote $q = (F_1, F_2)$ if $F_1 = \text{Re}(q)$ and $F_2 = \text{Im}(q)$. In local coordinate charts $(x, y)$ on $S_g$ in which $q = (dx + idy)^2$ the vertical measured foliation of $h^t(q)$ is determined by the
from $|dx + t \, dy|$; that is $h^t$ acts by the matrix

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

on $\mathcal{QT}_g$ [17]. The measure $\mu_g$ is invariant under the horocycle flow.

The relationship between the horocycle flow and earthquake flow implies the following result:

**Corollary 1.2.** The earthquake flow on $\mathcal{P}^1 \mathcal{M}_g$ is ergodic with respect to the Lebesgue measure class.

**Example.** A point $X \in \mathcal{T}_{g,n}$ is a complete hyperbolic surface of genus $g$ with $n$ punctures. Later in this paper, we only consider the case of $n = 0$. However, first we elaborate the simple part of the proof in the case of the once punctured torus where $g = n = 1$. In this case, $\dim(\mathcal{T}_{1,1}) = 2$. We remark that for $X, Y \in \mathcal{T}_{1,1}$, $Y = tw^t(X)$ for some $t \in \mathbb{R}$ if and only if $\ell_\lambda(X) = \ell_\lambda(Y)$. Hence there is a one-to-one correspondence between earthquake paths in $\mathcal{P}^1 \mathcal{T}_{1,1}$ and elements of $\mathcal{ML}_{1,1}$; a measured lamination $\lambda \in \mathcal{ML}_{1,1}$ corresponds to the locus $\{(\lambda, X) \in \mathcal{PT}_{1,1} \mid \ell_\lambda(X) = 1\}$. Recall that the space $\mathcal{ML}_{1,1}$ is homeomorphic to $\mathbb{R}^2$ as a piecewise integral linear manifold; the integral points in $\mathcal{ML}_{1,1}$ are exactly the integral multiples of simple closed curves on $S_{1,1}$.

On the other hand, $\text{Mod}_{1,1}$ acts naturally on both $\mathcal{ML}_{1,1}$ and the earthquake paths on $\mathcal{P}^1 \mathcal{T}_{1,1}$. This correspondence is mapping class group equivariant. Although $\mathcal{ML}_{1,1}$ has infinite volume, one can discuss the ergodic properties of a group acting on it. Namely, the action of the group $G$ on $\mathcal{ML}_{1,1}$ is ergodic if and only if for any $G$ invariant measurable subset $A \subset \mathcal{ML}_{1,1}$ either $\nu(A) = 0$, or $\nu(\mathcal{ML}_{1,1} - A) = 0$. It is known that the action of the mapping class group on $\mathcal{ML}_{1,1}$ is ergodic. This statement is equivalent to the ergodicity of the action of $SL_2(\mathbb{Z})$ on $\mathbb{R}^2$. It is easy to check that if $A \subset \mathcal{ML}_{1,1}$ has measure zero, then the measure of the set $\{(\lambda, X) \mid \ell_\lambda(X) = 1, \lambda \in A\}$ with respect to $\nu_{1,1}$ is zero.

Therefore, ergodicity of the action of the mapping class group on $\mathcal{ML}_{1,1}$ implies that the earthquake flow is ergodic. In general, the ergodicity of the Teichmüller horocycle flow shows that the action of the mapping class group on $\mathcal{ML}_{g,n}$ is ergodic [17]. However, when $\dim(\mathcal{T}_{g,n}) > 2$ this statement is weaker than the ergodicity of the earthquake flow.
The key tool in proving Corollary 1.2 in general is the work of Thurston and Bonahon [35], [2] on horocycle foliations and shear coordinates for the Teichmüller space and the space of measured foliations. As we will see later, this construction is a natural generalization of the Fenchel–Nielsen coordinates for the Teichmüller space.

1.2 Horocyclic foliation on a hyperbolic surface

By definition, for a maximal measured lamination \( \lambda \) on \( X \in T_g \), \( X - \lambda \) consists of finitely many hyperbolic ideal triangles. On the other hand, all hyperbolic ideal triangles are isomorphic. So one can use the canonical measured foliation on an ideal hyperbolic triangle by horocycles (as in Figure 1) to obtain a measured foliation \( F_\lambda(X) \) on \( X \). See Section 6 for more details.

From the construction, it is clear that \( i(\lambda, F_\lambda(X)) > 0 \); Moreover, \( F_\lambda(X) \in \mathcal{MF}_g(\lambda) \), where

\[ \mathcal{MF}_g(\lambda) = \{ \eta \mid \forall \gamma \ i(\gamma, \lambda) + i(\gamma, \eta) \neq 0 \}. \]

The key tool in proving our main result is the following:

**Theorem 1.3.** For any maximal measured lamination \( \lambda \),

\[ F_\lambda : T_g \to \mathcal{MF}_g(\lambda) \]

is a continuous and volume-preserving bijection such that

\[ \ell_\lambda(X) = i(\lambda, F_\lambda(X)). \quad (1.1) \]

Moreover, \( F_\lambda \) sends the earthquake flow for \( \lambda \) to the corresponding Teichmüller horocycle flow in a time-preserving fashion; namely,

\[ F_\lambda \left( tw^\ell_\lambda(X) \right) = \text{Im}(h^\ell(\lambda, F_\lambda(X))). \]

Recall that the pair \( (\lambda, F_\lambda(X)) \) uniquely determines a quadratic differential \( q = (\lambda, \eta) \) such that \( \text{Re}(q) = \lambda \) and \( \text{Im}(q) = F_\lambda(X) \) § 4.

For a discussion of natural volume forms on the Teichmüller space and the space of measured laminations see Section 2.
As the map $F_\lambda$ is defined for almost every $\lambda \in \mathcal{ML}_g$, we can define a measurable map $F : \mathcal{PT}_g \rightarrow \mathcal{QT}_g$, by

$$F(\lambda, X) = (\lambda, F_\lambda(X)).$$

By Theorem 1.3, this map sends the earthquake flow to the corresponding Teichmüller horocycle flow. On the other hand, by Equation (1.1), $\|F(\lambda, X)\| = \ell_\lambda(X)$. Hence the map $F$ descends to a map

$$F : \mathcal{P}^1 \mathcal{M}_g \rightarrow \mathcal{Q}^1 \mathcal{M}_g,$$

which will be denoted by the same letter. Then Theorem 1.3 implies that

$$F^*(\mu_g) = \nu_g.$$

Remark. We remark that a quadratic differential $q$ in the image of $F$ can only have simple zeros. Moreover, such a quadratic differential cannot have any imaginary saddle connections.

Let $\pi_P$ be the projection map $\pi_P : \mathcal{PT}_g \rightarrow \mathcal{T}_g$. There is a natural $\pi_Q : \mathcal{QT}_g \rightarrow \mathcal{T}_g$ sending a quadratic differential to the hyperbolic metric in the conformal class of its flat metric. Note that $\pi_P \neq \pi_Q \circ F$, and although $F_\lambda(X) = F_{t\lambda}(X)$, $F(t\lambda, X) \neq t \cdot F(\lambda, X)$.

Given $Y \in \mathcal{T}_g$ and $\lambda \in \mathcal{ML}_g$, let $\text{Ext}_i(Y)$ be the extremal length of $\lambda$ on $Y$ [1]. Then $i(\lambda, F_\lambda(X)) = \text{Ext}_i(\pi_Q(F(\lambda, X)))$. Hence we get

$$\ell_\lambda(X) = \text{Ext}_i(\pi_Q(F(\lambda, X))).$$

An important role in our approach is played by the space of transverse cocycles introduced by Bonahon [2] (Section 5). As we will explain later, given any maximal lamination $\lambda$, transverse cocycles on $\lambda$ parametrize both $\mathcal{T}_g$ and $\mathcal{ML}_g(\lambda)$.

1.3 Piecewise linear integral structures and induced measures

The crucial point in our discussion is that $\mathcal{ML}_g, \mathcal{QT}_g$, and the space of transverse cocycles on $\lambda$ all have natural piecewise integral linear structures. Namely, the transition functions between the natural charts are integral linear maps. In this paper, we investigate
the relationship between these piecewise linear structures. The foliation defined by

\[ E_\lambda = \{ q \in Q T_g \mid \text{Re}(q) = \lambda \} \subset Q M_g \]

inherits a piecewise linear structure from \( Q M_g \). We show the map \( e_\lambda : E_\lambda \to M F_g(\lambda) \) defined by \( e_\lambda(q) = \text{Im}(q) \) is a piecewise linear integral map.

The measures \( \mu_{Th}, \mu_g \) are in fact the measures induced by these structures. Moreover, the integral points have simple geometric characterization; the integral points in \( M L_g \) correspond to integral multicurves on \( S_g \).

Recall that \( \gamma \in M L_g \) is a multicurve if it can be written as \( \gamma = \sum_{i=1}^{k} c_i \gamma_i \) where \( c_i > 0 \), and \( \gamma_i \)'s are nonhomotopic, nontrivial, simple closed curves on \( S_g \); the multicurve \( \gamma \) is integral if \( c_i \in \mathbb{N} \).

**Remark.** As we will see later, the space of measured foliations carries a natural symplectic form. In fact, the map \( F_\lambda \) is a symplectomorphism, and the horocycle flow is the Hamiltonian flow of the intersection function \( g(\eta) = i(\lambda, \eta) \) on \( M F_g(\lambda) \). Nevertheless, since the space \( M F_g \) does not have a differentiable structure, we prefer to use the volume forms in order to avoid technical difficulties.

We will also discuss the construction of the invariant measure \( M_g \) for the earthquake flow on \( P^1 M_g \). It is known that the space \( Q^1 M_g \) has finite volume [16, 36]. By using the relationship between \( P^1 M_g \) and \( Q^1 M_g \), we obtain:

**Theorem 1.4.** The volume of the space \( Q^1 M_g \) is given by

\[
\text{Vol}(Q^1 M_g) = \int_{M_g} B(X) \, dX,
\]

where \( B(X) = \text{Vol}(\{ \lambda \mid \ell_\lambda(X) \leq 1 \}) \) with respect to the Thurston volume form \( \mu_{Th} \) on \( M L_g \). 

Using the method developed in [22] and [23] for integrating certain geometric functions with respect to the Weil–Petersson volume form, one can calculate the integral of \( B(X) \) over the moduli space explicitly. In fact, \( \text{Vol}(Q^1 M_g) \) can be written in terms of the intersection pairings of tautological classes on moduli spaces of surfaces with punctures. As a result, \( \text{Vol}_{\mu_g}(Q^1 M_g) \) is a rational multiple of \( \pi^{6g-6} \). See [24] for more details. The value of \( \text{Vol}(Q^1 M_g) \) arises in several problems related to billiards and dynamics of interval
exchange maps. Volumes of moduli spaces of holomorphic Abelian differentials have been calculated in [6]. For the calculation of volumes of moduli spaces of quadratic differentials see [7].

Remarks.

1. The key ideas used in this paper are similar to the work of Penner and Papadopoulos [30]. But here instead of working with a triangulation of a punctured surface, we work with measured laminations on compact surfaces and consider the ones that give rise to triangulations on the universal cover.

2. By work of Penner, the Weil–Petersson symplectic form on the Teichmüller space and the Thurston symplectic form on the space of measured laminations are related as follows. The Teichmüller space can be enlarged to the space \( \tilde{Y} \) consisting of all finite-area metrics on the surface, which have constant negative curvature \(-1\). There are natural embeddings of \( Y \) and \( \mathcal{MF}_g \) into \( \tilde{Y} \) and \( \tilde{Y} \) to form a completion \( \tilde{Y} \). Then the rescaled pulled-back of Weil–Petersson form on \( Y \) extends continuously to Thurston’s symplectic form. Also, in this picture the horocycle flow is the limit of the earthquake flow [29].

3. Let \( \pi \) be the projection map \( \pi : QT_g \rightarrow T_g \). The results of Hubbard and Masur [11] also induce a map \( A_\lambda : T_g \rightarrow \mathcal{MF}_g(\lambda) \) such that the quadratic differential \( q = (\lambda, A_\lambda(X)) \) satisfies \( \pi(q) = X \). The remark that the maps \( F_\lambda \) and \( A_\lambda \) are not the same. Also see Section 8 in [20].

4. Using the results of this paper, we prove an equidistribution result for the level sets of the lengths of simple closed curves in the moduli space [25]. □

1.4 Basic example

We remark that the map \( F_\lambda \) is defined for any maximal geodesic lamination. To exemplify the ideas used in the paper, we explain the special case of Theorem 1.3 for the lamination \( \lambda_p \) containing a pants decomposition \( P \). In Section 7, we study the map \( F_{\lambda_p}(X) \); In this case, one can explicitly calculate the Dehn–Thurston coordinates of \( F_{\lambda_p}(X) \) in terms of the Fenchel–Nielsen coordinates of \( X \in T_g \) with respect to \( P \).

We first briefly recall how the shear coordinates parameterize hyperbolic pairs of pants. Let \( \Sigma \) be an oriented pair of pants, and \( \partial \Sigma = \{ \alpha_1, \alpha_2, \alpha_3 \} \). Then the orientation of \( \Sigma \) induces an orientation on the boundary components. The pair of pants \( \Sigma \) is made of
gluing two ideal triangles as follows. Let

\[ \lambda_\Sigma = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \lambda_{1,2} \cup \lambda_{1,3} \cup \lambda_{2,3} \]

where \( \lambda_{i,j} \) is the simple geodesic spiraling around \( \alpha_i \) and \( \alpha_j \) in the positive direction. It is easy to see that \( \lambda \) is a maximal lamination; the complementary region of \( \lambda \) consists of two hyperbolic ideal triangles \( \Delta_1 \) and \( \Delta_2 \).

Let \( \Delta_1 \) and \( \Delta_2 \) be two ideal triangles containing the geodesic \( h \). In each ideal triangle \( \Delta \), there is a unique region whose boundary consists of three horocycles meeting tangentially at three points on \( \partial \Delta \). There is a canonical way to foliate the rest of \( \Delta \) by horocycles as in Figure 1.

Hence there are two canonical points \( p_{1,i,j} \) and \( p_{2,i,j} \) on the geodesic \( h = \lambda_{i,j} \) corresponding to the horocyclic foliations of \( \Delta_1 \), and \( \Delta_2 \). Let \( s_h(\Delta_1, \Delta_2) \in \mathbb{R} \) be the distance between \( p_{1,i,j} \) and \( p_{2,i,j} \) on \( h \).

More precisely, the shearing \( s_{i,j} \) along \( \lambda_{i,j} \) is the signed distance between \( p_{1,i,j} \) and \( p_{2,i,j} \). It is easy to check that \( e^{s_{i,j}} \) is equal to the cross ratio of the four points \( P_1, P_2, P_3 \) and \( P_4 \) on the boundary of the universal cover of the pair of pants \( \Sigma \) (see Figure 2). The
lengths of the three geodesic boundary components of $\Sigma$ are determined by

$$\ell(\alpha_i) = \sum_{j \neq i} s_{i,j}.$$  

Also, there are two canonical points $x_i$ and $y_i$ on $\alpha_i$ induced by the extension of the horocyclic foliations on $\Delta_1$ and $\Delta_2$.

2 Teichmüller Space and Measured Laminations

In this section, we briefly recall the background material on Teichmüller theory of Riemann surfaces and measured laminations.

2.1 Teichmüller space

A point in the Teichmüller space $T(S)$ is a complete hyperbolic surface $X$ equipped with a diffeomorphism $f : S \rightarrow X$. The map $f$ provides a marking on $X$ by $S$. Two marked surfaces $f : S \rightarrow X$ and $g : S \rightarrow Y$ define the same point in $T(S)$ if and only if $f \circ g^{-1} : Y \rightarrow X$ is isotopic to a conformal map. Let $\text{Mod}(S)$ denote the mapping class group of $S$, or the group of isotopy classes of orientation preserving self-homeomorphisms of $S$. The mapping class group $\text{Mod}_g = \text{Mod}(S_g)$ acts on $T_g$ by changing the marking. The quotient space

$$\mathcal{M}_g = \mathcal{M}(S_g) = T_g / \text{Mod}_g$$

is the moduli space of Riemann surfaces homeomorphic to $S_g$ [12]. The space $T_g$ is a finite-dimensional complex manifold equipped with the Weil–Petersson metric. The Weil–Petersson metric is a noncomplete Kähler metric, which is invariant under the action of the mapping class group.

2.2 The Fenchel–Nielsen coordinates

A pants decomposition of $S$ is a set of disjoint simple closed curves which decompose the surface into pairs of pants. Fix a system of pants decomposition of $S_g$, $\mathcal{P} = \{ \alpha_i \}_{i=1}^k$, where $k = 3g - 3$. For a marked hyperbolic surface $X \in T_g$, the Fenchel–Nielsen coordinates associated with $\mathcal{P}$, $\{ \ell_{\alpha_i}(X), \ldots, \ell_{\alpha_k}(X), \tau_{\alpha_i}(X), \ldots, \tau_{\alpha_k}(X) \}$, consists of the set of lengths of all
geodesics used in the decomposition and the set of the twisting parameters used to glue the pieces. We have an isomorphism

\[ T_g \cong \mathbb{R}_+^P \times \mathbb{R}^P \]

by the map

\[ X \rightarrow (\ell_{a_i}(X), \tau_{a_i}(X)). \tag{2.1} \]

By work of Wolpert, over the Teichmüller space the Weil–Petersson symplectic form induced by the Weil–Petersson metric has a simple form in Fenchel–Nielsen coordinates [38].

**Theorem 2.1** [37, 38]. The Weil–Petersson symplectic form is given by

\[ \omega_{wp} = \sum_{i=1}^{k} d\ell_{a_i} \wedge d\tau_{a_i}. \]

For more on the Weil–Petersson geometry of the Teichmüller space, see [12].

### 2.3 Measured laminations

A geodesic measured lamination \( \lambda \) consists of a closed subset of a hyperbolic surface \( X \) foliated by complete simple geodesics and a measure on every arc \( k \) transverse to \( \lambda \).

For understanding geodesic measured laminations, it is helpful to consider the lift to the universal cover \( \mathbb{H}^2 \) of \( X \). A directed geodesic is determined by a pair of points \( (x_1, x_2) \in S^\infty \times S^\infty - \Delta \), where \( \Delta \) is the diagonal \( \{(x, x)\} \). A geodesic without direction is a point on \( J = (S^\infty \times S^\infty - \Delta)/\mathbb{Z}_2 \), where \( \mathbb{Z}_2 \) acts by interchanging coordinates. Given a measured geodesic lamination \( \lambda \), the preimage of its underlying geodesic lamination \( A \subset D \) is decomposed as a union of geodesics of \( D \). Then geodesic laminations on two homeomorphic hyperbolic surfaces may be compared by passing to the circle at \( \infty \). So the notion of a measured lamination only depends on the topology of the surface \( X \).

The weak topology on measures induces the measure topology on the space of measured laminations; in other words, this topology is induced by the weak topology on the space of measured on a given arc, which is transverse to each lamination from an open subset of \( \mathcal{ML}_g \).
Given two closed curves $\gamma_1, \gamma_2$ on $S_g$, the intersection number $i(\gamma_1, \gamma_2)$ is the minimum number of points in which representatives of $\gamma_1$ and $\gamma_2$ must intersect. The intersection pairing extends to a continuous map

$$i : \mathcal{ML}_g \times \mathcal{ML}_g \to \mathbb{R}_+$$

See [10, 32] for more on the space of measured laminations and related topics.

A measured lamination $\lambda$ is filling if for every simple closed curve $\gamma$, $i(\gamma, \lambda) > 0$. If $\lambda$ is filling, then the complementary regions of $\lambda$ are ideal polygons. A measured lamination $\lambda$ is maximal if it is filling and all the complementary polygons are triangles.

### 2.4 Train tracks

The space of measured laminations $\mathcal{ML}_g$ is a piecewise linear manifold homeomorphic to $\mathbb{R}^{6g-6}$. Train tracks were introduced by Thurston as a powerful technical device for understanding measured laminations. Roughly speaking train tracks are induced by squeezing almost parallel strands of a very long simple closed geodesic to simple arcs on a hyperbolic surface. A train track $\tau$ on a surface $S$ is a finite closed 1 manifold $\tau \subset S$ with switches embedded on $S$ such that

- **T-1:** $\tau$ is $C^1$ away from its switches and has tangent vectors at every point, and
- **T-2:** for each component $R$ of $S - \tau$, the double of $R$ along the interiors of edges of $\partial(R)$ has negative Euler characteristic.

The vertices (or switches) of a train track are the points where three or more smooth arcs come together. Each edge of $\tau$ is a smooth path with a well-defined tangent vector; That is all edges at a given vertex are tangent. The inward-pointing tangent of an edge divides the branches that are incident to a vertex into incoming and outgoing branches.

A lamination $\gamma$ on $S$ is carried by $\tau$ if there is a differentiable map $f : S \to S$ homotopic to the identity taking $\gamma$ to $\tau$ such that the restriction of $dF$ to a tangent line of $\gamma$ is nonsingular. Every geodesic lamination is carried by some train track. When a lamination $\lambda$ carried by the train track $\tau$ has an invariant measure $\mu$, then the carrying map defines a counting measure $\mu(b)$ to each branch line $b : \mu(b)$ is just the transverse measure of the leaves of $\lambda$ collapsed to a point on $b$. At a switch, the sum of the entering numbers equals the sum of the exiting numbers.
The integral piecewise linear on $\mathcal{ML}_g$ is induced by train tracks as follows. Let $E(\tau)$ be the set of measures on train track $\tau$; more precisely, $u \in E(\tau)$ is an assignment of positive real numbers on the edges of the train track satisfying the switch conditions of the form,

$$
\sum_{\text{incoming } e_i} u(e_i) = \sum_{\text{outgoing } e_j} u(e_j).
$$

A train track $\tau$ is recurrent if it supports a positive measure with $u(b) > 0$ for every edge $b$. Also, let $W(\tau)$ be the vector space of all real edge weight systems on $\tau$ satisfying the switch conditions; here $u(e_i)$ need not be positive.

By Thurston's work [32] any maximal recurrent train track $\tau$, $E(\tau)$ gives rise to an open set $U(\tau)$ in the space of measured laminations; such a train track defines a chart on the manifold $\mathcal{ML}_g$. Hence we can identify $W(\tau)$ with the tangent space of $\mathcal{ML}_g$ at a point $u \in E(\tau)$ [10].

For any train track $\tau$, the integral points in $E(\tau)$ are in one-to-one correspondence with the set of integral multicurves in $U(\tau) \subset \mathcal{ML}_g$. The natural volume form on $E(\tau)$ defines a mapping class group invariant volume form $\mu_{Th}$ in the Lebesgue measure class on $\mathcal{ML}_g(S)$. We remark that $\mathcal{ML}_g$ does not have a natural differentiable structure [32].

### 2.5 Thurston symplectic form on $\mathcal{ML}_g$

In fact, the volume form on $\mathcal{ML}_g$ is induced by a mapping class group invariant two form $\omega$ as follows. The train track $\tau$ is generic if all switches are trivalent; at each vertex there are two incoming and one outgoing edge. Let $\tau$ be a generic train track. For $u_1, u_2 \in W(\tau)$, the symplectic pairing is defined by

$$
\omega(u_1, u_2) = \frac{1}{2} \left( \sum_v u_1(e_1) u_2(e_2) - u_1(e_2) u_2(e_1) \right),
$$

(2.2)

where the sum is over all vertices of the train track, $e_1$ and $e_2$ are the two incoming branches at $v$ such that $e_1$ is on the right side of the common tangent vector. This form defines a skew symmetric bilinear form on $W(\tau)$.

**Lemma 2.2.** If the train track $\tau$ is maximal, then Thurston's form $\omega$, defined by Equation (2.2), is nondegenerate. Therefore it defines a symplectic form on the piecewise linear manifold $\mathcal{ML}_g$. \qed
See [10] for a proof and also a discussion of the relationship between the homology intersection paring on $H_1(S, \mathbb{R})$ and Thurston’s intersection pairing.

2.6 Combinatorial type of measured laminations and train tracks

Each component of $S - \lambda$ is a region bounded by closed geodesics and infinite geodesics, and it can be doubled along its boundary to give a complete hyperbolic surface which has finite area. We say a filling measured lamination $\lambda$ is of type $a = (a_1 \ldots a_k)$ iff $S - \lambda$ consists of ideal polygons with $a_1 \ldots a_k$ sides.

By extending the measured lamination $\lambda$ to a foliation with isolated singularities on the complement, we have $\sum_{i=1}^{k} (a_i - 1) = 8g - 8$ (see [32]). Similarly, each component of the complement of a filling train track $\tau$ is a nonpunctured or once-punctured cusped polygon of negative Euler index. We say a train track $\tau$ is of type $a = (a_1, \ldots, a_k)$, iff $S - \tau$ consists of $k$, polygons with $a_1 \ldots a_k$ sides. Every measured lamination of type $a = (a_1, \ldots, a_k)$ can be carried by a train track of type $a$.

**Lemma 2.3.** For any recurrent filling train track $\tau$ of type $(a_1, \ldots, a_k)$, we have

$$\dim(E(\tau)) = 2g + k - 1$$\text{ if } \tau \text{ is orientable},

$$\dim(E(\tau)) = 2g + k - 2$$\text{ if } \tau \text{ is not orientable}.

□

As a result, a generic measured lamination $\lambda \in \mathcal{ML}_g$ cuts $S_g$ into ideal triangles.

**Lemma 2.4.** Almost every $\lambda \in \mathcal{ML}_g$ is maximal. □

**Proof.** First, note that for a fixed simple closed curve $\gamma$,

$$\dim(\{\lambda \mid i(\gamma, \lambda) = 0\}) \leq 6g - 8.$$

Therefore, the set of nonfilling geodesic laminations is a union of countably many lower-dimensional subspaces of $\mathcal{ML}_g$. Hence almost every measured lamination is filling.

Since there are only finitely many types of measure laminations on $S_g$, it is enough to show that the measure of the set of measured laminations of type $(a_1, \ldots, a_k)$ is zero unless $k = 4g - 4$ in which case we will get $a_1 = \cdots = a_k = 3$. Note that $a_i - 1 \geq 2$, so $\sum_{i=1}^{k} (a_i - 1) = 8g - 8$ implies that $k \leq 4g - 4$. Also, there are countably many train tracks
of type \((a_1, \ldots, a_k)\) on \(S_g\). Hence \(\dim(E(\tau)) < 6g - 6\) unless \(k = 4g - 4\). So if \(k \neq 4g - 4\), the set of measured laminations of type \((a_1, \ldots, a_k)\) is a countable union of lower-dimensional linear subsets of \(\mathcal{ML}_g\), so it has measure zero.

\[\blacksquare\]

2.7 Length functions and the earthquake flow

For any simple closed geodesic \(\alpha\) on \(X \in \mathcal{T}_g\) and \(t \in \mathbb{R}\), we can deform the hyperbolic structure as follows. We cut the surface along \(\alpha\), turn left-hand side of \(\alpha\) in the positive direction the distance \(t\) and reglue back. Let us denote the new surface by \(\text{tw}^t_\alpha(X)\). As \(t\) varies, the resulting continuous path in the Teichmüller space is the Fenchel–Nielsen deformation of \(X\) along \(\alpha\). For \(t = \ell_\alpha(X)\), we have

\[\text{tw}^t_\alpha(X) = \phi_\alpha(X),\]

where \(\phi_\alpha \in \text{Mod}(S_g)\) is a right Dehn twist about \(\alpha\). Notice that the notion of right and left only depend on the orientation of the underlying surface.

The twist deformation was introduced by Fenchel and Nielsen. Later, Wolpert studied the connection with the Weil–Petersson symplectic form. Thurston generalized the twisting from curves to laminations. Also, he showed any two points in Teichmüller space are related by a unique right earthquake [33]. This result was used by Kerckhoff in his solution to the Nielsen realization problem [13].

By Wolpert’s result (Theorem 2.1), the vector field generated by twisting around \(\alpha\) is symplectically dual to the exact one form \(d\ell_\alpha\) [37]. Here \(\ell_\alpha\) denotes the geodesic length of closed curve \(\alpha\) in \(X\). On the other hand, both the length function and the twisting deformation are extended by homogeneity and continuity on \(\mathcal{ML}_g\) [14]. More precisely, there is a unique continuous map

\[L : \mathcal{ML}_g \times \mathcal{T}_g \to \mathbb{R}_,\]

such that for any simple closed curve \(\alpha\), \(L(\alpha, X) = \ell_\alpha(X)\), and \(L(t \cdot \lambda, X) = t \cdot L(\lambda, X)\). Then \(\ell_\lambda(X) = L(\lambda, X)\) is the geodesic length of the measured lamination \(\lambda\) on \(X\). Similarly, the earthquake deformation is the extension of the twist deformation for a general lamination such that

\[\text{tw}^t_{r,\eta}(X) = \text{tw}^t_{\eta^{-1}}(X)\].
In other words, $\text{tw}_t^\lambda$ is the limit of time $t$ twist deformation of any sequence $\{r_i \gamma_i\}$ converging to $\lambda$ in $\mathcal{ML}_g$.

Recall that any smooth function $H$ on a symplectic manifold $(M, \omega)$ gives rise to a vector field $X_H$ satisfying

$$\omega(X_H, Y) = dH(Y).$$

It is easy to check that the Hamiltonian flow of the function $H$ generated by the vector field $X_H$ preserves both $H$ and $\omega$.

**Theorem 2.5.** The earthquake flow along $\lambda$, $\text{tw}_t^\lambda$ is the the Hamiltonian flow of the length function $\ell_\lambda$. Therefore, given $\lambda \in \mathcal{ML}_g$, and $t \in \mathbb{R}$, the map $\text{tw}_t^\lambda : T_g \rightarrow T_g$ is a symplectomorphism. \hfill \Box

See [14] for more details.

### 3 Measures for the Earthquake Flow

In this section, we study finite earthquake flow invariant measures on $\mathcal{P}^1 \mathcal{M}_g$. For $X \in T_g$, consider $B_X = \{ \lambda \mid \ell_\lambda(X) \leq 1 \} \subset \mathcal{ML}_g$, and let $B(X) = \text{Vol}(B_X)$. We prove:

**Theorem 3.1.** There exists a finite measure $\nu_g$ in the Lebesgue measure class on $\mathcal{P}^1 \mathcal{M}_g$ which is invariant under the earthquake flow. This measure projects to the volume form given by $B(X) \cdot \mu_{wp}$ on $\mathcal{M}_g$. \hfill \Box

**Proof.** Consider the Thurston volume form $\mu_{Th}$ on $\mathcal{ML}_g$, and the Weil–Petersson volume form $\mu_{wp}$ on the Teichmüller space. We know:

- Fixing $\lambda$ and $t \in \mathbb{R}$, the map $\text{tw}_t^\lambda : T_g \rightarrow T_g$ is the Hamiltonian flow of the length function of $\lambda$. Therefore, this map preserves the Weil–Petersson volume. Now consider the flow $T_\lambda$ defined by

$$T_\lambda^t(X) = \text{tw}_t^{\ell_\lambda(X)}(X).$$
By the definition, $\ell_i(X) = \ell_i(T_{t \lambda}^i(X))$, and $T_{t \lambda}^i(X) = T_{t \lambda}^i(X)$. Also, if $\ell_i(X) = 1$, then $T_{t \lambda}^i(X) = tw_{t \lambda}^i(X)$. Moreover, given $t \in \mathbb{R}$ and $\lambda \in \mathcal{ML}_g$, the map

$$T_{t \lambda}^i : \mathcal{T}_g \to \mathcal{T}_g$$

is volume preserving.

- The Thurston volume form is mapping-class-group invariant.

Therefore, $\mu_{Th} \times \mu_{wp}$ gives rise to a mapping-class-group invariant measure on $\mathcal{ML}_g \times \mathcal{T}_g$. This measure is invariant under the flow $T^i(\lambda, X) = (\lambda, T_{t \lambda}^i(X))$. By the definition, $T^i(s \lambda, X) = (s \lambda, T^i(X))$. The length function

$$\mathcal{L} : \mathcal{ML}_g \times \mathcal{T}_g \to \mathbb{R},$$

defined by $\mathcal{L}(\lambda, X) = \ell_i(X)$ is invariant under the earthquake flow and the action of the mapping class group. Now $\nu_g$ is the measure induced by $\mu_{Th} \times \mu_{wp}$ on $\mathcal{P}^1 \mathcal{M}_g$. In other words, any point $X \in \mathcal{T}_g$ define a measure $\nu_X$ on $\mathcal{P} \mathcal{ML}_g$ such that for $A \subset \mathcal{P} \mathcal{ML}_g$, we have

$$\nu_X(A) = \text{Vol}([\lambda \mid [\lambda] \in A, \ell_i(X) \leq 1]),$$

with respect to the Thurston volume form on $\mathcal{ML}_g$. The Weil–Petersson volume form on $\mathcal{M}_g$ and the family $\{\nu_X\}_{X \in \mathcal{T}_g}$ of measures on $\mathcal{P} \mathcal{ML}_g$ combine to give the invariant measure $\nu_g$. Since $\nu_X(\mathcal{P} \mathcal{ML}_g) = B(X)$, the results of [21] imply that

$$\text{Vol}(\mathcal{P}^1 \mathcal{M}_g) = \int_{\mathcal{M}_g} B(X) \, dX < \infty.$$  

In fact,

$$B(X) = \lim_{L \to \infty} \frac{b(X, L)}{L^{5g-6}},$$

where $b(X, L)$ is the number of multigeodesics of length $\leq L$ on $X$.

**Remark.** In general, any locally finite mapping-class-group invariant measure on $\mathcal{ML}_g$ gives rise to an earthquake flow invariant measure on $\mathcal{P}^1 \mathcal{M}_g$. For example, we can use the discrete measure supported on the orbit of a multicurve in $\mathcal{ML}_g$. For any multicurve
γ and $L > 0$, one can define an ergodic, earthquake flow invariant measure $v_\gamma$ supported on

$$H_\gamma(L) = \{ (\alpha, X) | \alpha \in \text{Mod} \cdot \gamma, \ell_\alpha(X) = L \} \subset \mathcal{ML}_g \times T_g.$$  

It is easy to check that $H_\gamma(L)$ projects to a closed subset of $\mathcal{M}_g$.  

4 Quadratic Differentials and Measured Foliations

4.1 Moduli space of quadratic differentials

A holomorphic quadratic differential $q$ is a tensor $q = f(z) dz^2$ with a locally defined holomorphic function $f(z)$. For $g \geq 2$ we define the space $QT_g \to T_g$ to be the bundle of quadratic differentials over the Teichmüller space. Then a point $(x, q)$ in $Q\mathcal{M}_g = QT_g/\text{Mod}_g$ consists of a compact Riemann surface of genus $g$ with a holomorphic quadratic differential $q$ on $X$. Although the value at a point $x \in X$ of a quadratic differential depends on the local coordinates, the zero set of $q$ is well defined. As a result, there is a natural stratification of the space $QT_g$ by the multiplicities of zeros of $q$.

Define $QT_g(a_1, \ldots, a_m) \subset QT_g$ to be the subset consisting of pairs $(X, q)$ of holomorphic quadratic differentials on $X$ with $m$ zeros with multiplicities $(a_1, \ldots, a_m)$, and $Q\mathcal{M}_g(a_1, \ldots, a_k) = QT_g(a_1, \ldots, a_k)/\text{Mod}_g$. The Gauss–Bonnet formula implies that $4g(S) - 4 = \sum_{i=1}^{m} a_i$. Then

$$Q\mathcal{M}_g = \bigsqcup_{(a_1, \ldots, a_m)} Q\mathcal{M}_g(a_1, \ldots, a_k).$$

It is known that each $Q\mathcal{M}_g(a_1, \ldots, a_k)$ is an orbifold of dimension $4g(S) - 4 + 2k$. In particular, $\dim(Q\mathcal{M}_g(1, \ldots, 1)) = \dim(Q\mathcal{M}_g)$.

One way to understand this moduli space is by studying the period coordinates. Note that any quadratic differential defines a flat metric with cone type singularities on the surface. If a quadratic differential $q = f(z) (dz)^2$ is holomorphic and nonzero at $z = z_0$, then on a neighborhood of $z_0$ the quadratic differential is given by $q = (d\sqrt{f})^2$, where $w(z) = \int_{z_0}^{z} \sqrt{f(z)} dz$. We just need to choose the neighborhood small enough so that a single-valued branch of the function $\sqrt{f}$ can be chosen. On a small chart, which contains a singularity, coordinate $w$ can be chosen in such way that $q = z^k dw^2$, where $k$ is the order of the singularity. Note that there is an ambiguity of coordinate change $w(z) = \pm w(z) + a$ with an
arbitrary complex constant \( a \). See [31] and for more details. A quadratic differential defines a metric on the surface such that outside singular set the metric has zero curvature. So the image of a geodesic in any chart is a straight line. A saddle connection is a geodesic segment, which joins a pair of singular points without passing through one in its interior. In general, a geodesic segment \( e \) joining two zeros of a quadratic differential \( q = \phi dz^2 \) determines a complex number \( \text{hol}_q(e) \) (after choosing a branch of \( \phi^{1/2} \) and an orientation of \( e \)) by

\[
\text{hol}_q(e) = \Re(\text{hol}_q(e)) + \Im(\text{hol}_q(e)),
\]

where

\[
\Re(\text{hol}_q(e)) = \int_e \Re(\phi^{1/2}),
\]

and

\[
\Im(\text{hol}_q(e)) = \int_e \Im(\phi^{1/2}).
\]

In fact, one can choose saddle connections \( e_1, \ldots, e_{6g-6} \) such that \( \{\text{hol}(e_i)\} \) give rise to a local chart for a neighborhood of \( q \) [18]. The period coordinates gives \( QT_g \) the structure of a complex manifold. It also defines a measure \( \mu_g \) by the volume form

\[
\prod_{i=1}^{6g-6} d\Re(\text{hol}(e_i)) \wedge d\Im(\text{hol}(e_i)).
\]

We need the following simple but essential points later:

- Both

\[
\frac{\Re(\text{hol}_\phi(e))}{\Im(\text{hol}_\phi(e))}, \quad \text{and} \quad \Re(\text{hol}(e_i)) \times \Im(\text{hol}(e_i))
\]

are independent of the choice of \( \sqrt[\phi]{} \), and the orientation of \( e \). So they only depend on the quadratic differential \( q \) and the arc \( e \).
• The horocycle flow has a simple form in the holonomy coordinates; for any quadratic differential \( q \), \( h^t(q) \) is determined by

\[
\text{Re}(\text{hol}_{h^t} q(e)) = \text{Re}(\text{hol}_q(e)) + t \, \text{Im}(\text{hol}_q(e)), \tag{4.1}
\]

and

\[
\text{Im}(\text{hol}_{h^t} q(e)) = \text{Im}(\text{hol}_q(e)).
\]

### 4.2 Measured foliations

A measured foliation is a foliation of the surface \( S \) with a transverse measure and only finitely many singularities similar to the singularities of holomorphic quadratic differentials. Let \( \mathcal{MF}(S) \) be the set of equivalence classes of measured foliations on \( S \) with generalized saddle singularities (three prongs or more), where the equivalence relation is generated by isotopy and Whitehead moves (i.e. collapsing saddle connections). For any curve \( \gamma \), \( i(F, \gamma) \) is the transverse length of \( \gamma \) by \( F \). Two measured foliations \( F_1 \) and \( F_2 \) are equivalent if \( i(F_1, \beta) = i(F_2, \beta) \) for all classes \( \beta \). The space \( \mathcal{MF}_g \) a piecewise manifold of dimension \( 6g - 6 \). See [33].

### 4.3 Real and imaginary foliations

A holomorphic quadratic differential \( q \) on \( X \) determines two measured foliations \( \text{Re}(q) \) and \( \text{Im}(q) \) such that near a nonsingular point \( p \) with canonical coordinate \( z = x + iy \), horizontal leaf segments are parallel to the \( x \)-axis and the transverse measure on \( \text{Re}(q) \) is defined by integration of \( |dy| \), while vertical leaf segments are parallel to the \( y \)-axis with transverse measure defined by integrating \( |dx| \). The foliations \( \text{Re}(q) \) and \( \text{Im}(q) \) have singularities of the same type at the zeros of \( q \).

Therefore we get a map \( \mathcal{P} : QT_g \to \mathcal{MF}_g \times \mathcal{MF}_g - \Delta \) by

\[
\mathcal{P}(q) = (\text{Re}(q), \text{Im}(q)). \tag{4.2}
\]

Here \( \Delta = \{(\lambda, \eta) \mid \exists \gamma \in \mathcal{MF}_g, i(\gamma, \lambda) + i(\gamma, \eta) = 0\} \).

**Theorem 4.1.** The map \( \mathcal{P} \) is a mapping class group equivariant homeomorphism. \( \square \)
For the proof see [9]. As a corollary, if for every simple closed curve $\gamma$ on $S_g \ i(\gamma, \lambda) + i(\gamma, \eta) > 0$ then one can find representatives of $\lambda$ and $\eta$ such that they are transverse apart from their singularities and their singularities coincide and are of the same order. See also [27] for a different proof of this corollary without using quadratic differentials.

For any saddle connection $e$ joining two zeros of the quadratic differential $q$,

$$|\text{Im}(\text{hol}_q(e))| = i(e, \text{Re}(q)).$$

(4.3)

Here $i(e, \text{Re}(q))$ is the transverse measure of the arc $e$ with respect to the measured foliation $\text{Re}(q)$.

**Lemma 4.2.** Let $q \in QT_g$. Then the measured foliation $\text{Re}(q) \in \mathcal{MF}_g$ is integral if and only if $\text{Im}(\text{hol}_q(e)) \in \mathbb{Z}$ for any geodesic segment $e$ joining two zeros of the quadratic differential $q$. □

### 4.4 Piecewise linear structure of $QT_g$

The holonomy map gives rise to a piecewise linear structure on $QT_g(1, 1, \ldots, 1)$. On the other hand, $\mathcal{ML}_g \times \mathcal{ML}_g$ has a natural piecewise linear structure induced by train track charts of $\mathcal{ML}_g$. We show that these two piecewise linear structure are the same.

Fix $q_0 \in QT_g(1, \ldots, 1)$. Let $\Sigma$ be a triangulation of $S_g$ by geodesic saddle connections of $q_0 = (\lambda, \eta_0)$ such that the set of vertices is equal $\{ p \mid q_0(p) = 0\}$ [19]. We can choose $U_{q_0} \subset QM_g$ such that any $q \in U_{q_0}$ has a corresponding triangle by geodesic edges.

Consider the null-gon track $\tau$ dual to $\Sigma$, as in Figure 3, in each component of $S - \Sigma$ there is one triangle of $\tau$. This is closely related to the Ribbon graph dual to $\Sigma$. See also [29]. Note that $\tau$ has exactly $4g - 4$ complementary components which are once punctured null-gons. So $\tau$ is not a train track (see condition T-2 in Section 2). We will remove some of the edges of $\tau$ to obtain a train track $\tilde{\tau}$ on $S$ carrying $\lambda = \text{Re}(q_0)$.

First note that the construction of $\tau$ imposes a condition on the weights of the edges on $\tau$ corresponding to the lamination $\lambda$.

Let $\Delta$ be a triangle in $\Sigma$ with edges $e_1^\Delta, e_2^\Delta,$ and $e_3^\Delta$ then there are permutations $\{i_1, i_2, i_3\} = \{j_1, j_2, j_3\} = \{1, 2, 3\}$ such that

$$|\text{Im}(\text{hol}_{q_0}(e_{i_1}^\Delta))| = |\text{Im}(\text{hol}_{q_0}(e_{i_2}^\Delta))| + |\text{Im}(\text{hol}_{q_0}(e_{i_3}^\Delta))|,$$

$$|\text{Re}(\text{hol}_{q_0}(e_{j_1}^\Delta))| = |\text{Re}(\text{hol}_{q_0}(e_{j_2}^\Delta))| + |\text{Re}(\text{hol}_{q_0}(e_{j_3}^\Delta))|.$$
Now we define a train track \( \tilde{\tau} \subset \tau \), obtained by removing the edge corresponding to \( e^{\Delta}_{i_1} \); that is

\[
\tilde{\tau} = \tau - \bigcup_{\Delta \in \Sigma} e^{\Delta}_{i_1}.
\]

(4.4)

Then \( \lambda \) is carried by \( \tilde{\tau} \). It is easy to check that \( \tilde{\tau} \) satisfies both conditions T-1, and T-2, and hence it is a train track (Section 2). In fact, if \( q_0 \in QM_1(1, \ldots, 1) \), then \( S - \tilde{\tau} \) consists of only trigons. This train track carries the lamination \( \text{Re}(q) \) for any \( q \) in a neighborhood around \( q_0 \).

Then the piecewise linear structures of \( U_{q_0} \subset QM_g \) and \( P(U_{q_0}) \) coincide:

\[ \text{Lemma 4.3.} \] The map \( P : QT_g \to MF_g \times MF_g - \Delta \) is piecewise linear on \( QM_g(1, 1, \ldots, 1) \subset QM_g. \] □

\[ \text{Remark.} \] We remark that a foliation \( \eta \) is determined by its intersection numbers with the edges of the triangulation \( \Delta \) (see [26]). Note that in the case we consider, all the singularities of \( \eta \) are vertices of \( \Delta \) (see Figure 3).

Given edge \( e_0 \) of \( \tilde{\tau} \) define

\[
e_0^0 = \text{Re}(\text{hol}_q(e_0))\text{sgn}(\text{Im}(\text{hol}_q(e_0))) \cdot \text{Re}(\text{hol}_q(e_0)) \in \mathbb{R},
\]

(4.5)

where \( q = (\lambda, \eta) \). We remark that these weights also satisfy the switch conditions; for any vertex \( \nu \) \( e_1^0 + e_2^0 = e_3^0 \). Hence \( e^0 \) define an element in \( W(\tilde{\tau}) \). See (I) in Section 5.
4.5 Volume form on $\mathcal{QT}_g$

As we mentioned before, the period map

$$\phi : \mathcal{QT}_g \rightarrow \mathbb{C}^{3g-3}$$

gives rise to a local coordinate system on $\mathcal{QT}_g$. In terms of this local coordinate system the measure $\mu_g$ is simply the pull back of the Euclidean Lebesgue measure on $\mathbb{C}^{3g-3}$. We say a quadratic differential $q \in \mathcal{QT}_g$ is integral if for any edge $e$ joining two zeros of $q$, we have $\text{Re}(\text{hol}_q(e)) \in \mathbb{Z}$, and $\text{Im}(\text{hol}_q(e)) \in \mathbb{Z}$. Now we use the following observation on volumes induced by piecewise linear integral structures. Assume that $X$ and $Y$ both have piecewise linear integral structures. Let $f : X \rightarrow Y$ be a piecewise linear homeomorphism such that

- V-1: $f(t \cdot x) = t \cdot f(x)$, and
- V-2: $x \in X$ is integral if and only if $f(x)$ is integral in $Y$.

Then $f$ is volume preserving.

The map $\mathcal{P}$ sends the integral points of $\mathcal{QT}_g$ in holonomy coordinates to the integral points of $\mathcal{MF}_g \times \mathcal{MF}_g$. Also, note that $\mathbb{R}_+$ acts on both $\mathcal{QT}_g$ and $\mathcal{MF}_g \times \mathcal{MF}_g$ such that

$$\mathcal{P}(t \cdot q) = t \cdot \mathcal{P}(q),$$

and

$$\|q\| = i(\text{Re}(q), \text{Im}(q)).$$

By Lemma 4.3, the map $\mathcal{P}$ satisfies conditions V-1 and V-2. Hence, we have:

**Corollary 4.4.** The map $\mathcal{P}$ (defined by 4.2) is volume preserving.

5 Transverse Cocycles

In this section, we recall some background on transverse cocycles for a measured lamination $\lambda$, and their role in understanding measured laminations transverse to $\lambda$. 
5.1 Transverse cocycles and train tracks

A transverse cocycle \( \sigma \) for a geodesic lamination \( \lambda \) is a finitely additive signed transverse measure for \( \lambda \). More precisely, \( \sigma \) assigns a real number to each arc transverse to \( \lambda \) such that

- **C-1:** \( \sigma(k) = \sigma(k_1) + \sigma(k_2) \) if \( k = k_1 \cup k_2 \) and \( k_1 \) and \( k_2 \) are disjoint,
- **C-2:** \( \sigma(k_1) = \sigma(k_2) \) for transversely homotopic arcs \( k_1 \) and \( k_2 \); that is \( \sigma \) is invariant under homotopy of arcs transverse to \( \lambda \).

Hence any transverse measure on \( \lambda \) is a transverse cocycle for \( \lambda \). Let \( H(\lambda, \mathbb{R}) \) be the vector space of transverse cocycles on \( \lambda \). See [4] and [3] for more details.

**Example.** Let \( \eta = \lambda \cup \gamma \), where \( \gamma \) is a simple closed geodesic on \( X \epsilon \mathcal{M}_{1,1} \), and \( \lambda \) is the geodesic lamination whose ends spiral around \( \gamma \). Choose two transverse arcs to \( \eta \) such that \( k_1 \cup \gamma = \phi \) and meets \( \lambda \) once, and \( k_2 \) meets \( \gamma \) in one point. One can check that for any \( a, b \in \mathbb{R} \) there is a unique \( \sigma \in H(\lambda, \mathbb{R}) \) such that \( \sigma(k_1) = a \) and \( \sigma(k_2) = b \). Therefore \( \dim(H(\lambda, \mathbb{R})) = 2 \).

On the other hand, the dimension of the space of transverse measures for \( \eta \) is one.

**Remark.** In fact, transverse cocycles on \( \lambda \) correspond to transverse invariant Hölder distributions on \( \lambda \) [3, 4].

In general, by work of Bonahon \( H(\lambda, \mathbb{R}) \) has finite dimension and similarly to the space of measured laminations, can be understood using train tracks carrying \( \lambda \). If \( \lambda \) is carried by a train track \( \tau \), any transverse cycle \( \sigma \), induces a set of real weights on the edges of \( \tau \) satisfying the switch condition which defines a natural map \( i : H(\lambda, \mathbb{R}) \to W(\tau) \). Conversely, in the generic case a transverse cocycle for \( \lambda \) is actually determined by a weight system on the train track carrying \( \lambda \).

**Theorem 5.1.** [2, 3] If \( \lambda \) is a maximal geodesic lamination carried by a train track \( \tau \), then the map

\[
i : H(\lambda, \mathbb{R}) \to W(\tau)
\]

is a bijection. Moreover, the Thurston symplectic form defined on \( W(\tau) \) induces a pairing \( w \) on \( H(\lambda, \mathbb{R}) \), which does not depend on \( \tau \).

\[ H_+ (\lambda) = \{ x \mid w(x, \eta) > 0 \text{ for every transverse measure } \eta \text{ supported on } \lambda \}. \quad (5.1) \]

Remarks.

1. Note that if \( x \in H_+ (\lambda) \) then for any transverse measure \( \eta \) supported on \( \lambda \), \( x + \eta \in H_+ (\lambda) \). In particular for any \( t > 0 \), \( x + t \lambda \in H_+ (\lambda) \).
2. There is a close relationship between measured foliations and measured laminations on a surface. In fact, it is easy to see that by straightening the leaves of a measured foliation one can obtain a measured lamination. Conversely, for \( \lambda \in \mathcal{ML}_g \), carried by a train track \( \tau \), we can define a measured foliation \( \tilde{\lambda} \) on a neighborhood of \( \tau \) induced by \( \lambda \). Then by collapsing each region of \( X - \tau \) into a spine, we can extend this measured foliation. The measured foliation \( \tilde{\lambda} \) is well defined up to equivalent classes of \( \mathcal{MF}_g \). Also given \( \lambda, \eta \in \mathcal{MF}(S) \), \( i(\lambda, \eta) = i(\tilde{\lambda}, \tilde{\eta}) \), and

\[ \mathcal{MF}_g(\lambda) \cong \mathcal{ML}_g(\lambda). \]

Therefore, measured laminations and measured foliations are essentially the same [15]. We say a measured foliation \( \mathcal{F} \) is integral if the corresponding measured lamination is a multicurve with integral weights.

The map \( \mathcal{P} \) defined in (4.2) induces a map

\[ QT_g \to \mathcal{ML}_g \times \mathcal{ML}_g - \Delta, \]

which is denoted by the same letter. In this paper, we work with both measured laminations and measured foliations.

5.2 Shear coordinates for measured foliations

Let \( \lambda \) be a maximal geodesic lamination, and \( \eta \in \mathcal{ML}_g(\lambda) \); that is for any \( \gamma \in \mathcal{ML}_g \),

\[ i(\gamma, \lambda) + i(\gamma, \eta) > 0. \]
Then one can choose transverse measured foliations in the class of $\eta$ and $\lambda$. Recall that two measured foliations are transverse if their singular sets coincide with the same number of prongs and the leaves are transverse apart from the singular set. We sketch how $H_+(\lambda)$ provides a coordinate system for $\MF_g(\lambda)$ using shearing.

Let $k$ be an arc transverse to $\lambda$; in particular, $i(k, \eta) \neq 0$. To define the transverse cocycle $I_\lambda(\eta)$, we modify $\sigma_1(k) = i(k, \eta)$ into a transverse cocycle $\sigma(k)$.

Let $\alpha$ be the part of a leaf of $\lambda$ joining $r_1$ to $r_0$, and $\beta$ be a part of a leaf of $\eta$ joining $r_2$ to $r_0$ such that $\alpha \cap \beta = r_0$. Let $v_\alpha$ and $v_\beta$ denote the tangent vectors at $r_0$ to the paths $\alpha$ (parameterized from $r_1$ to $r_0$) and $\beta$ (parameterized from $r_0$ to $r_2$). Define $\epsilon(\alpha, \beta) = 1$ if the orientation of the pair $\langle v_\alpha, v_\beta \rangle$ is the same as the orientation of the underlying surface $S$; otherwise define $\epsilon(\alpha, \beta) = -1$.

Fix an arc $k$ transverse to $\lambda$ joining $r_1$ to $r_2$. First, assume that there is a point $r_0$ on the surface such that $r_2$ and $r_0$ lie on a leaf of $\eta$ and $r_1$ and $r_0$ lie on a leaf of $\lambda$, and the path $\alpha, \beta$ is in the homotopy class of $k$.

**Definition 5.2.** Define $\sigma = I_\lambda(\eta)$, the $\lambda$ shear of $\eta$ along the transverse arc $k$, by

$$\sigma(k) = \epsilon(\alpha, \beta) \cdot i(k, \eta) = \epsilon(\alpha, \beta) \cdot i(\beta, \eta). \tag{5.2}$$

In general, for a path $\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i, \ldots, \alpha_n$ (joining $r_1$ to $r_2$) in the homotopy class of $k$, where $\alpha_i$ is part of a leaf of $\lambda$ and $\beta_i$ is a part of a leaf of $\eta$, define

$$\sigma(k) = \sum_{i=1}^{n} \epsilon(\alpha_i, \beta_i) i(\alpha_i, \eta).$$

One can check that $\sigma(k)$ does not depend on the choice of the path. Therefore, $\sigma$ satisfies conditions C–1 and C–2 (see the definition of transverse cocycles in the beginning of this section).

**Lemma 5.3.** For any $\eta \in \mathcal{ML}_g(\lambda)$, $\sigma$ defines a transverse cocycle $I_\lambda(\eta) \in H(\lambda, \mathbb{R}).$ \hfill \square

Note that when $\lambda$ is a maximal lamination, $\dim(\mathcal{ML}_g(\lambda)) = \dim(H(\lambda)) = 6g - 6$. See [35] for a similar way to parameterize elements of $\mathcal{ML}_g(\lambda)$ by their intersections with edges of a train track carrying $\tau$. 
Next, we will study the properties of the map

\[ I_\lambda : \mathcal{ML}_g(\lambda) \to H(\lambda) \]

defined by (5.2). Recall that the space \( H(\lambda) \) is equipped with a volume form induced by a train track carrying \( \lambda \). We use the relationship between the shear and holonomy coordinates to show that the map \( I_\lambda \) is volume preserving. The following two observations will be useful in studying the basic properties of the map \( I_\lambda \):

\textbf{I. Shear coordinates and holonomy map.} We now describe how \( I_\lambda(\eta) \) can be computed in terms of holonomy coordinates of the quadratic differential \( q = (\lambda, \eta) \). Let \( k \) be a geodesic saddle connection joining two singular points of \( q \), and let \( \sigma_q(k) \) denote the \( \lambda \) shear of \( \eta \) along \( k \). Then by definition, for \( \sigma = I_\lambda(\eta) \), we have \( |\sigma(k)| = |\text{Re}(\text{hol}_q(k))| = i(k, \eta) \). Moreover, one can determine the real number \( \sigma(k) \) in terms of holonomies along saddle connections; more precisely it is easy to check that \( \sigma(k) \) is determined by

\[ \frac{\sigma(k)}{|\text{Im}(\text{hol}_q(k))|} = \frac{\text{Re}(\text{hol}_q(k))}{\text{Im}(\text{hol}_q(k))}. \]  

(5.3)

Note that as the lamination \( \lambda \) is maximal, \( \text{Im}(\text{hol}_q(k)) \neq 0 \). In general, for any saddle connection \( k \),

\[ \frac{\text{Im}(\text{hol}_q(k))}{\text{Re}(\text{hol}_q(k))} \]

is independent of the orientation of \( k \) and the choice of \( \sqrt{q} \). So in terms of this notation, if the path \( \langle \alpha, \beta \rangle \) is in the homotopy class of \( k \), then

\[ \epsilon(\alpha, \beta) = \text{sgn} \left( \frac{\text{Im}(\text{hol}_q(k))}{\text{Re}(\text{hol}_q(k))} \right). \]

Compare with (4.5).

\textbf{II. Shear coordinates in terms of train track weights.} One can calculate the shear coordinates in terms of the weights induced on the train track \( \tau \) carrying \( \lambda \). Let \( \mathcal{F} \) be the measured foliation transverse to \( \lambda \) representing \( \eta \in \mathcal{ML}_g(\lambda) \). For any branch \( b \) of the train track \( \tau \), there are two (not necessarily distinct) triangles of \( S - \tau \) adjacent to \( b \). There is a unique three-pronged singularity of \( \mathcal{F} \) in each of these triangles, and there is a
leaf emanating from the singular point that hits $b$. Now we assign the real number of shearing or algebraic distance between the hitting points on each side of $b$ [28].

Also, there is a related interpretation in terms of the tangential measure induced by $F$ on $\tau$. Recall that the set of tangential measures on any train track $\tau$ approximation of $\lambda$ gives rise to a subset $MF_g(\lambda)$. More precisely, let $V^*(\tau)$ is the set of tangential measures on $\tau$. Then there is a continuous injection

$$i_\tau : V = V^*(\tau)/ \sim \to MF_g(\lambda),$$

where $\sim$ is a linear equivalence relation on $V^*(\tau)$ (see Section 3.4 in [10] for more details). Moreover, if the train tack $\tau$ is complete, i.e. $\tau$ is of combinatorial type $(3, \ldots, 3)$, then the image of the map in (5.4) is open in $MF_g(\lambda)$.

In terms of this notation, the shear along a branch $b$ is one-half of the alternating sum of the total tangential measures assigned to the sides of the polygon obtained by gluing the two triangles of $S - \tau$ adjacent to $b$ together. See Section 9 in [35] and [32]. Hence the map

$$I_\lambda \circ i_\tau : V \to H(\lambda, \mathbb{R})$$

is linear.

5.3 Transverse cocycles and intersection pairings

We can now calculate the intersection pairing $\omega(\lambda, I_\lambda(\eta))$ defined by (2, 2). If $\Delta$ is a geodesic triangle on $q = (\lambda, \eta)$, then the area of the triangle $\Delta$ with respect to the flat structure of $q$ can be calculated by

$$\text{Area}_q(\Delta) = \frac{1}{2}(\text{Re}(\text{hol}_q(e_1)) \times \text{Im}(\text{hol}_q(e_2)) - \text{Im}(\text{hol}_q(e_1)) \times \text{Re}(\text{hol}_q(e_2))),$$

where $e_1$ and $e_2$ are two edges of $\Delta$ so that the orientation induced by $\langle e_1, e_2 \rangle$ is the same as the orientation of the underlying surface. As a result, we obtain the following lemma:

**Lemma 5.4.** Let $\lambda$ be a maximal measured lamination, and $I_\lambda(\eta)$ be the transverse cocycle corresponding to $\eta \in ML_g(\lambda)$. Then we have

$$i(\lambda, \eta) = \|q\| = \omega(I_\lambda(\eta), \lambda) > 0,$$
where $q$ is the quadratic differential defined by $\lambda$ and $\eta$, and $\omega$ is the Thurston symplectic form (2.2).

Sketch of the proof. Note that the pairing $\omega(\lambda, I_\lambda(\eta))$ is independent of $\tau$ [2]. So we can choose a suitable train track $\tau$ carrying $\lambda$ such that we can use Equation (5.5). Recall that any quadratic differential $q$ admits a triangulation whose vertices are the singularities $q$ and whose edges are saddle connections.

Let $\tilde{\tau}$ be the dual of the triangulation carrying $\lambda = \text{Re}(q)$ as in Section 4.4. See Figure 4. We recall that $\|q\| = \text{Area}(q)$ with respect to the affine charts defined by $q$. We calculate the area of the triangle $\Delta_v$ corresponding to the vertex $v$ of the train track $\tilde{\tau}$ with respect to the quadratic differential $q$ defined by $\eta$ and $\lambda$ as follows. For any edge $e$ of $\tilde{\tau}$, and $\gamma \in H(\lambda, \mathbb{R})$, let $e(\gamma)$ be the weight of $e$ in the corresponding element of $W(\tilde{\tau})$. By Equations (5.3), (4.3), and (5.5), the area of $\Delta_v$ equals

$$\frac{1}{2} \left( e_1^{\Delta_v}(\lambda) \cdot e_2^{\Delta_v}(I_\lambda(\eta)) - e_1^{\Delta_v}(I_\lambda(\eta)) \cdot e_2^{\Delta_v}(\lambda) \right),$$

where $e_1^{\Delta_v}$ and $e_2^{\Delta_v}$ are incoming edges adjacent to the vertex $v$ of $\tilde{\tau}$. Hence,

$$\text{Area}(q) = \sum_v \text{Area}_q(\Delta_v) = \omega(\lambda, I_\lambda(\eta)).$$

Therefore, by (5.1)

$$I_\lambda(\mathcal{ML}_g(\lambda)) \subset H_+(\lambda).$$
We remark that $H_+(\lambda)$ has a linear structure. Also, a point $\sigma \in H_+(\lambda)$ is \emph{integral} if for every transverse arc $k$, $\sigma(k) \in \mathbb{Z}$.

\textbf{Remark.} Let $q_0$ be the quadratic differential defined by $(\lambda, \eta_0)$, and let $\Sigma$ be a triangulation by saddle connections. Let $\tilde{\tau}$ be the train track defined in Section 4 carrying $\lambda$ (see Figure 3). Then for any $\eta$ close to $\eta_0$, the weights of $|I_\lambda(\eta)|$ on the edge $e$ of $\tilde{\tau}$ is the same as the real part of the holonomy induced by $q = (\lambda, \eta)$ on $e$ (see Equation (4.5)). \hfill $\square$

6 \textbf{From $\mathcal{ML}_g$ to Holomorphic Quadratic Differentials}

In this section, we use the work of Bonahon [2] and Thurston [35] on shearing to prove the main result of the paper.

6.1 \textbf{Horocycle foliation $F_\lambda(X)$ corresponding to a maximal lamination on $X \in \mathcal{T}_g$}

Let $\tilde{X}$ be the universal cover of $X$. Then $\tilde{\lambda}$, the preimage of a lamination $\lambda$, gives rise to a collection of geodesics of $\tilde{X}$ invariant by the action of $\pi_1(X)$. If $\lambda$ is maximal, then every lift of a leaf of $\lambda$ is on the boundary of at least one ideal triangle.

On each ideal triangle in the universal cover, there is a well-defined partial foliation (that is, a foliation supported on a subsurface of that triangle) whose leaves are connected pieces of horocycles centered at the vertices of these triangles. See Figure 1. The end points of each leaf are on the edges of that triangle making right angles with the edges. The nonfoliated region is a central triangle, which is bounded by three of these horocyclic arcs meeting tangentially at their endpoints. Now the foliation $F_\lambda(X)$ associated with a metric $X$ on $\tilde{X}$ arises from partially foliating each ideal hyperbolic triangle of $D - \tilde{\lambda}$ and then joining these parts to cover $D$. This partial foliation induces a foliation of full support on the ideal triangle, by collapsing the nonfoliated region onto a tripod.

Since the tangent line field to the geodesic lamination $\lambda$ is Lipschitz [2], this foliation can be extended to $D$. Also, the geodesic length along leaves of $\lambda$ gives rise to a measure on arcs transverse to this foliation [35]. Since the construction is $\pi_1$ invariant, it defines a measured foliation $F_\lambda(X)$ on $X = D/\pi_1(X)$. See [35] for more details.

6.2 \textbf{Shearing along edges of triangles}

We sketch how $X \in \mathcal{T}_g$ gives rise to a transverse cocycle $G_\lambda(X) \in H(\lambda, \mathbb{R})$. Let $k$ be an arc transverse to $\lambda$. Consider the universal cover of $X$, and $\tilde{k}$ be the preimage of $k$. Now let $\tilde{k}$
be one of the preimages of \( k \) joining \( p_i \) to \( p_j \). If \( \lambda \) is maximal, each complementary region of \( \lambda \) is an ideal triangle. Let \( p_i \in \Delta_i \). Here although \( \Delta_1 \) and \( \Delta_2 \) do not have an edge in common, given two leaves \( g \) and \( h \) of the geodesic lamination \( \tilde{\lambda} \), the horocyclic measured foliation gives rise to an isometry between \( \theta^{gh} : g \to h \). Define \( s(\Delta_1, \Delta_2) \in \mathbb{R} \) such that the isometry \( \theta^{gh} \) corresponds to \( t \to t - s(\Delta_1, \Delta_2) \) \[2\]. Then \( \sigma = G_{\lambda}(X) \) is defined by

\[
\sigma(k) = s(\Delta_1, \Delta_2). \tag{6.1}
\]

We remark that \( s(\Delta_1, \Delta_2) \) is a generalization of the shear \( s_{i,j} \) discussed in Section 1; if \( \Delta_1 \) and \( \Delta_2 \) are adjacent then \( s(\Delta_1, \Delta_2) \) is determined by the cross ratio of the corresponding quadrilateral.

Moreover, the map \( X \to G_{\lambda}(X) \) satisfies the following properties \[2\]:

- Given two metrics \( X_1, X_2 \in T_g \) and a maximal geodesic lamination \( \lambda \), there is a unique shear map with fault locus \( \lambda \), which sends \( X_1 \) to \( X_2 \).
- If \( X_1 \) is transformed into \( X_2 \) by a shear map with fault locus \( \lambda \), then transverse cocycle measuring the shifts to the left of this shear map is exactly \( G_{\lambda}(X_1) - G_{\lambda}(X_2) \).

Using the definitions of \( G_{\lambda} \) and \( F_{\lambda} \), it is easy to check that:

**Proposition 6.1.** The map \( G_{\lambda} \)

\[
G_{\lambda} : T_g \to H_+(\lambda), \quad X \to \sigma(X),
\]

is given by

\[
G_{\lambda}(X) = I_\lambda(F_{\lambda}(X)).
\]

**Proof.** Note that the transverse measure of \( F_{\lambda}(X) \) is induced by the geodesic length along the leaves of \( \lambda \). Let \( \tilde{\sigma} = I_\lambda(F_{\lambda}(X)) \). Definition (5.2) for \( \eta = F_{\lambda}(X) \) implies that for any transverse arc \( k \) joining \( p_1 \in \Delta_1 \) to \( p_2 \in \Delta_2 \) as in Figure 5. Also compare with Figure 2. We have

\[
\tilde{\sigma}(k) = i(F_{\lambda}(X), k) = d_h(q_1, q_2),
\]
Fig. 5. Transverse measured foliation $F_\lambda(X)$.

where $d_h(q_1, q_2)$ is the signed geodesic distance between $q_1, q_2 \in h$. Now Equation (6.1) implies that $\tilde{\sigma}(k) = \sigma(k) = G_\lambda(X)(k)$. 

We use the following result of Bonahon and Sözen [5]:

**Theorem 6.2.** The map $G_\lambda$ is a symplectomorphism such that

$$\ell_\lambda(X) = \omega(G_\lambda(X), \lambda).$$

As a result,

- the map $G_\lambda$ is volume preserving;
- $G_\lambda(tw^+_\lambda(X)) = G_\lambda(X) + t \cdot \lambda \in H_+(\lambda)$, (6.2)

where $\lambda$ is the transverse cocycle corresponding to the transverse measure of the measured lamination $\lambda \in \mathcal{ML}_g$. See the remark after Theorem 5.1.

We remark that given a maximal lamination $\lambda$, the map $G_\lambda$ induces a linear structure on $T_{\overline{g}}$; the rays with respect to this linear structure correspond to the shear paths for transverse cocycles on $\lambda$.

On the other hand, we have
Theorem 6.3. The map $I_\lambda : \mathcal{M}L_g(\lambda) \to H_+(\lambda)$ is a volume-preserving homeomorphism. Moreover, if $q_t = h^t(\lambda, \eta_0) = (\lambda, \eta_t)$ then we have

$$I_\lambda(\eta_t) = I_\lambda(\eta_0) + t \cdot \lambda \in H_+(\lambda).$$  \hspace{1cm} (6.3) \hspace{1cm} \blacksquare$$

Sketch of the proof. Recall that

$$\dim(H_+(\lambda)) = \dim(\mathcal{M}L_g(\lambda)),$$

also the map $I_\lambda$ is piecewise linear. By the preceding theorem the map $G_\lambda = I_\lambda \circ F_\lambda$ is surjective. Hence the map $I_\lambda$ is also surjective. So we have to prove that the map $I_\lambda$ is one to one.

Assume that for $\eta_1, \eta_2 \in \mathcal{M}L_g(\lambda)$, $I_\lambda(\eta_1) = I_\lambda(\eta_2)$. Then there exists a complete train track $\tau$ carrying $\lambda$ such that both $\eta_1$ and $\eta_2$ are transverse to $\tau$. Let $V = V^*(\tau)/\sim$ be the linear quotient of the space of tangential measures on $\tau$ (see Section 3.4 in [10]). Then by the discussion in Section 5, the map induced by $I_\lambda: V \to H_+(\lambda, \mathbb{R})$ is a linear map. Since $\dim(V) = \dim(H(\lambda, \mathbb{R}_+))$, the map $I_\lambda|_V$ is one to one, and $\eta_1 = \eta_2$.

To show that the map $I_\lambda : \mathcal{M}L_g(\lambda) \to H_+(\lambda)$ is volume preserving, note that $I_\lambda(t \cdot \eta) = t \cdot I_\lambda(\eta)$. Also, by Lemma 4.2, $I_\lambda(\eta)$ is an integral point in $H_+(\lambda)$ if and only if $\eta$ is an integral multicurve. Therefore, $I_\lambda$ satisfies conditions V-1 and V-2 in Section 4. Since the volume forms on $\mathcal{M}L_g$ and $H_+(\lambda)$ are induced by their piecewise linear structure, we conclude the map $I_\lambda$ is volume preserving.

Also, Equation (6.3) is an immediate consequence of Equations (4.1) and (5.3). \hspace{1cm} \blacksquare

Remark. By a result of Papadopoulos [27], the horocycle flow corresponding to $\lambda$

$$h^t_\lambda : \mathcal{MF}_g(\lambda) \to \mathcal{MF}_g(\lambda)$$

defined by

$$h^t_\lambda(\eta) = \text{Im}(h^t(\lambda, \eta))$$

is the Hamiltonian flow of the intersection function $i(\lambda, \cdot)$ on $\mathcal{MF}_g(\lambda)$. One can show that the map $F_\lambda$ is a symplectomorphism. Note that as $\ell_\lambda(X) = i(F_\lambda(X), \lambda)$, Theorem 2.5 implies that the map $F_\lambda$ sends the earthquake path along $\lambda$ to the horocycle path around $\lambda$. \hspace{1cm} \blacksquare
Proof of Theorem 1.3. Consider the diagram

\[
\begin{align*}
T_g & \xrightarrow{G_\lambda} H_+(\lambda) \xrightarrow{I_\lambda^{-1}} \mathcal{ML}_g(\lambda).
\end{align*}
\]

The map \( F_\lambda = I_\lambda^{-1} \circ G_\lambda \) is volume preserving since \( G_\lambda \) and \( I_\lambda \) are both volume preserving. Also, by Equations (6.3) and (6.2), the map \( F_\lambda \) sends the earthquake path along \( \lambda \) to the horocycle path along \( \lambda \). □

We remark that for any maximal measured lamination \( \lambda \), \( F_\lambda(X) \in \mathcal{ML}_g(\lambda) \). Hence the pair \( (\lambda, F_\lambda(X)) \) defines a holomorphic quadratic differential. Consider the map

\[
F : \mathcal{ML}_g \times T_g \rightarrow QM_g
\]

\[
(\lambda, X) \rightarrow P^{-1}(\lambda, F_\lambda(X)).
\]

Then we have:

1. By Lemma 2.4, the map \( F \) is defined almost everywhere.
2. Therefore by Corollary ??, \( F \) is a measure-preserving map conjugating the earthquake and horocycle flow. Moreover, we have \( \ell_\lambda(X) = \|F(\lambda, X)\| = 1 \).

□

Remark. Let \( \mathcal{G} \subset \mathcal{ML}_g \) be the subset of maximal measured laminations on \( S_g \). The map \( F \) as defined in this paper is a bijection from \( \mathcal{G} \times T_g \) to

\[
\{ q \mid q \text{ does not have any imaginary saddle connections} \} \subset QM_g(1, 1, \ldots, 1).
\]

One can check that this map sends the stretch path along \( \lambda \) ([35]) to the Teichmüller geodesic flow in \( QM_g \). □

We remark that the map \( F \) does not extend continuously to \( \mathcal{ML}_g \times T_g \). To prove this claim, consider a lamination \( \lambda \) on \( X \in T_g \) such that one of the complementary regions of \( X - \lambda \) is a nonregular quadrilateral. In other words, the cross ratio of the four points \( P_1, P_2, P_3, \) and \( P_4 \) in the universal cover of \( X \) equals \( r \neq 1 \). Then there are two sequences \( \{\lambda^1_i\} \) and \( \{\lambda^2_i\} \) of maximal measured laminations such that the Hausdorff limit of \( \{\lambda^j_i\}_{i=1}^\infty \) contains \( d_j \) (see Figure 6). Let \( \eta^j = \lim_{i \to \infty} F(\lambda^j_i, X) \). Using basic hyperbolic trigonometry, one can check that if the quadrilateral \( P_1, P_2, P_3, P_4 \) is not regular, then \( c_1 \neq c_2 \), and therefore we have \( \eta^1 \neq \eta^2 \).
For $j = 1, 2$, let $\lambda_j = \lambda \cup d_j$. Then both $\lambda_1$ and $\lambda_2$ are maximal laminations (without transverse measures). Note that $F_{\lambda_j}(X)$ defines a measured foliation. However, $F_{\lambda_1}(X) \neq F_{\lambda_2}(X)$. Hence

$$\lim_{i \to \infty} F_{\lambda_1}(X) \neq \lim_{i \to \infty} F_{\lambda_2}(X).$$

7 Basic Example

In this section, we consider the case where $\lambda = \lambda_P$ is a maximal geodesic lamination containing a pants decomposition $P = \{\gamma_1, \ldots, \gamma_{3g-3}\}$ of $X$, defined in Section 1 by

$$\lambda_P = \bigcup_{\Sigma \in P} \lambda_\Sigma.$$

The relation between the geometry of $X \in T_g$, the transverse cocycle $G_\lambda(X)$, and the corresponding measured foliation $F_\lambda(X)$ is easier to understand in this special case.

7.1 Transverse cocycles supported on $\lambda_P$

The geodesic lamination $\lambda_P$ is carried by the train track $\tau_P$ defined such that its restriction to the pair of pants $\Sigma$ is of the form $\tau_\Sigma$ in Figure 7.

Note that the train track $\tau_P$ is of type $(3, 3, \ldots, 3)$. Therefore $\dim(E(\tau_P)) = 6g - 6$, and $H(\lambda_P, \mathbb{R}) \cong \mathbb{R}^{6g-6}$. 

Fig. 6. Decomposition of a non-regular quadrilateral.
Let $x \in W(\tau)$ be a system of weights satisfying the switch conditions on $\tau_\mathcal{P}$ (see Section 2). For simplicity let $\gamma_i$ denote the element of $W(\tau)$ representing $\gamma_i$. In Figure 7, $x_\alpha, \beta$ is the weight of the line joining curve $\alpha$ to $\beta$. Let $B_i(x)$ denote the weight of the edge $r_i$, and $A_i(x) = \omega(x, \gamma_i)$.

If $\gamma_i, \gamma_m, \gamma_n \in \mathcal{P}$ bound a pair of pants, then

$$A_i(x) = x_{\gamma_i, \gamma_m} + x_{\gamma_i, \gamma_n}.$$

It is easy to check that the transverse cocycle $x$ is uniquely determined by the set of numbers

$$(A_i(x), B_i(x))_{i=1}^{3g-3}.$$

See Theorem 5.1. The space $H_+(\lambda)$ consists of elements in $x \in H(\lambda, \mathbb{R})$ where for every $1 \leq i \leq 3g-3$, $A_i(x) > 0$. Hence, we have

$$H_+(\lambda) \cong \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}.$$

### 7.2 Measured laminations transverse to $\lambda_\mathcal{P}$

For $\lambda = \lambda_\mathcal{P}$

$$\mathcal{ML}_g(\lambda) = \{ \eta | \forall 1 \leq i \leq 3g - 3, i(\eta, \gamma_i) > 0 \}.$$
Consider the Dehn–Thurston parameterization of $\mathcal{ML}_g(\lambda)$ defined by

$$DT_P : \mathcal{ML}_g(\lambda) \to (\mathbb{R}_+ \times \mathbb{R})^{3g-3}$$

$$\eta \mapsto (m_i, t_i)_{i=1}^{3g-3},$$

(7.1)

where $m_j = i(\eta, \alpha_j) \in \mathbb{R}_+$ is the intersection number of $\eta$ and $\alpha_j$, and $t_j = \text{tw}(\eta, \alpha_j) \in \mathbb{R}$ is the twisting number of $\eta$ around $\alpha_j$. The Dehn–Thurston theorem asserts that these parameters uniquely determine $\eta$. See [10] for more details.

Then given $\eta \in \mathcal{ML}_g(\lambda)$, $I_\lambda(\eta)$ is the unique element $x_\eta \in H_+(\lambda, \mathbb{R})$ such that

$$A_i(x_\eta) = i(\eta, \gamma_i),$$

and

$$B_i(x_\eta) = \text{tw}(\eta, \gamma_i).$$

7.3 Connection with Fenchel–Nielsen coordinates

For any $\gamma_i \in \mathcal{P}$,

$$\ell_{\gamma_i}(X) = w(\lambda, G_{\lambda, \lambda}(X)) = i(\gamma_i, F_{\lambda, \lambda});$$

also we can read off the twisting parameter $\tau_i(X)$ around $\gamma$ from the values of $G_{\lambda, \lambda}(X)(k)$, where $k$ is an arc joining two points in two different pairs of pants containing $\gamma_i$. See Section 2.

Given $X \in T_g$, consider the restriction of the measured foliation $\eta = F_i(X)$ to a pair of pants $\Sigma$ where $\partial \Sigma = \{\gamma_i, \gamma_n, \gamma_n\} \subset \lambda_P$. There are two possible cases for $\Sigma$ (see Figure...
8). Let $x_i = i(\eta, \gamma_i)$. In case (a), $x_i > x_m + x_n$; case (b) happens when $(x_i, x_m, x_n)$ satisfies $|x_i - x_m| \leq x_n \leq x_i + x_m$ (see Section 4 of [16]). Also, see [8] (Exposé 8) for more on the map $F_\lambda$ in this case.

Let $(\ell_\gamma(X), \tau_i(X))_{i=1}^{3g-3}$ be the Fenchel–Nielson coordinates of $X \in T_g$ with respect to $\mathcal{P}$ (as in (2.1)). Then the map

$$F_\lambda : T_g \to \mathcal{ML}_g(\lambda) \cong (\mathbb{R}_+ \times \mathbb{R})^{3g-3}$$

is given by

$$X \to DT^{-1}_{\mathcal{P}}(\ell_\gamma(X), \tilde{\tau}_i(X))_{i=1}^{3g-3}$$

where

$$\tilde{\tau}_i(X) = \tau_i(X) + g_i(\ell_\gamma(X), \ldots, \ell_{\gamma_{3g-3}}(X)). \quad (7.2)$$

Using basic trigonometry, the function $g_i$ can be calculated explicitly. However, we do not need the explicit calculation to show that $F_\lambda$ is volume preserving. By Equation (7.2)

$$\bigwedge_{i=1}^{3g-3} d\ell_i \wedge d\tilde{\tau}_i = \bigwedge_{i=1}^{3g-3} d\ell_i \wedge d\tau_i.$$

Therefore, Theorem 2.1 implies that the map $F_\lambda$ is volume preserving.

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