Theorem: If $G \subseteq \text{PSL}_2(\mathbb{R})$ contains no elliptic elements then it is either elementary or discrete.

Proof: Let $\Gamma$ be hyperbolic $h = \begin{bmatrix} u & 0 \\ 0 & \frac{1}{u} \end{bmatrix}$, in order to prove $\Gamma$ is discrete, enough to show $\{gn\} \to \Gamma_0 \to \mathbb{H}$.

(why?) Let $G_n = \left[ \begin{array}{cc} a_n & b_n \\ c_n & d_n \end{array} \right]$ $a_n d_n - b_n c_n = 1$.

\[ \begin{array}{c} f_n \to 1, \\ \text{tr}(h g_n h^{-1} h_n) \to 2, \\ (2 - b_n c_n (u - \frac{1}{u})^2) \to 2. \end{array} \]

Why? $b_n c_n < 0$. Assume $d_n = 0$.

\[ \begin{array}{c} f_n = \left[ \begin{array}{cc} A_n & B_n \\ C_n & D_n \end{array} \right], \\ A_n D_n - B_n C_n = 1. \end{array} \]

\[ \begin{align*} \text{tr}[h, f_n] &= 2 - B_n C_n (u - \frac{1}{u})^2 \\ &\to 2. \end{align*} \]

$\Rightarrow n$ large $b_n c_n \to 0$.

$\Rightarrow 1 \Rightarrow$ for large $n > N$ $b_n c_n = 0$ $\Rightarrow h, g_n$ have a common fixed pt

For any hyperbolic element $h$; $h$ and $g_n$ have a fixed pt in $\mathbb{H}$.

We know $\exists h_1, h_2, h_3 \in \Gamma$ hyperbolic no two of which have a common fixed pt. Then for $n$ large enough $g_n$ has a common fixed pt with $h_1, h_2, h_3$.

**In fact, $\langle g, h \rangle$ non-elementary**

\[ \begin{align*} g_n &= \text{id.} \Rightarrow \\ \frac{\text{Sinh} \left( \frac{1}{2} \sqrt{d(zi, \bar{z})} \right)}{\text{Sinh} \left( \frac{1}{2} \sqrt{h_i(zi, \bar{z})} \right)} &\geq 1. \end{align*} \]