5.3. Sphere preserving transformations

In Chapter 2 we read a paper of Carathéodory proving that transformations of the plane that map circles to circles are extended Möbius transformations, i.e. linear fractional transformations in \( z \) or \( \bar{z} \), and hence are compositions of inversions and similarities. In this spirit we prove the following theorem of Möbius, which assumes the continuity of the mapping.

**Theorem 5.6.** Let \( f : U \to f(U) \) be a continuous 1-1 mapping defined on an open set in \( \mathbb{R}^n \), and suppose that \( f \) maps (pieces of) planes and spheres in \( U \) to (pieces of) planes and spheres in \( f(U) \) (not necessarily respectively). Then \( f \) is a composition of similarities and inversions.

**Proof.** For any \( x \in U \) choose \( x_0 \neq x \) in \( U \) and a ball \( B_r(x_0) \) such that the closed ball \( \bar{B}_r(x_0) = B_r(x_0) \cup S_r(x_0) \subset U \) but \( x \notin \bar{B}_r(x_0) \). Let \( y_0 = f(x_0) \), and let \( S_r(y_0) \) be any sphere about \( y_0 \). Let \( g \) and \( h \) be inversions in \( S_r(x_0) \) and \( S_r(y_0) \) respectively. Then \( h \circ f \circ g \) is defined on the exterior of \( \bar{B}_r(x_0) \), and the image of a hyperplane lying in this exterior is a hyperplane. Thus, considering intersections, \( h \circ f \circ g \) maps lines to lines. Also parallel lines \( l_1 \) and \( l_2 \) are mapped to parallel lines, even if the plane of the lines meets \( \bar{B}_r(x_0) \), for there exist non-parallel planes \( \pi_1 \) and \( \pi_2 \) containing \( l_1 \) and \( l_2 \) respectively, but not meeting \( \bar{B}_r(x_0) \), and \( l = \pi_1 \cap \pi_2 \) is parallel to both \( l_1 \) and \( l_2 \).

Now \( x \notin \bar{B}_r(x_0) \); therefore \( x \) is in the domain of \( h \circ f \circ g \). Let \( T_x : \mathbb{R}^n \to \mathbb{R}^n \) be translation by \( x \), i.e. the vector \( v \in \mathbb{R}^n \) is mapped to the vector \( v + x \). Then, setting \( y = (h \circ f \circ g)(x) \), we see that

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\varphi = T_{-y} \circ h \circ f \circ g \circ T_x
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