The goal of this supplement is to give a “direct” proof of the following proposition. Along the way we define lower semi-continuity and see that any lower semi-continuous function on a closed segment attains its minimum.

**Proposition.** If \( f : [a, b] \rightarrow \mathbb{R} \) is continuous, then it is also uniformly continuous.

**Proof.** Fix \( \varepsilon > 0 \). Then for every \( x_0 \in \text{dom } f \) there exists \( \delta \) such that \( |x - x_0| < \delta \) implies \( |f(x) - f(x_0)| < \varepsilon \) (it will be important later that this inequality is strict).

We shall consider the set of all such deltas \( S(x_0, \varepsilon) \) and take “the largest such \( \delta \)” - namely \( \delta(x_0) = \sup\{\delta \in S(x_0, \varepsilon), x_0 - a, b - x_0\} \).

This is a function of \( x_0 \). If we could show that it has a positive minimum \( \delta \), then we could take that as a \( \delta \) in the definition of uniform continuity and be done (check that you understand why this is true!). But why should \( \delta(x_0) \) have a minimum?

Well, we know that continuous functions on segments attain their minimums, so if \( \delta(x_0) \) were continuous, we would have \( \delta = \delta(x) > 0 \) for some minimizing \( x \in [a, b] \) and be done. However, \( \delta(x) \) is not continuous.

Indeed, consider \( f(x) = \sin x \) for \( 100 \leq x \leq 0 \) and \( 0 < x \leq -100 \). For \( \varepsilon = 1 \) at \( x = 0 \) we have \( \delta_1(0) = \pi/2 \) but for any \( x \) slightly positive we have \( \delta_1(x) = 3\pi/2 - 2x \).

Perhaps the problem is that when defining \( \delta(x) \) we have made the inequality in \( |f(x) - f(x_0)| < \varepsilon \) strict?

But if we make the inequality non-strict then the same example shows \( \delta_1 \) jumping from 100 to \( 3\pi/2 - 2x \).

Fortunately not all is lost. If we keep the inequality strict, then the value of \( \delta(x) \) can only jump up, but not down. This is called lower semi-continuous.

**Definition 0.1.** A function \( f \) is lower-semi continuous if for every \( \varepsilon > 0 \) exists \( \delta > 0 \) such that whenever \( |x - x_0| < \delta \) we get \( f(x) > f(x_0) - \varepsilon \).

The other half of the continuity definition – \( f(x) < f(x_0) + \varepsilon \) – is missing (it has the name of – what else? – upper semi-continuity).
It turns out lower semi-continuity is sufficient to ensure attainment of minima on closed segments.

**Proposition.** If \( f : [a, b] \to \mathbb{R} \) is lower semi-continuous then there exists \( s \in [a, b] \) such that \( f(x) \geq f(s) \) for all \( x \) in \([a, b]\).

**Proof.** The proof repeats the one for continuous \( f \). First, we show \( f \) is bounded below.

Indeed, if \( f \) is not bounded below, we build \( s_n \) with \( f(s_n) < -n \). Pick a subsequence converging to \( s \) and see that for \( \varepsilon = 1 \) there is \( \delta \) such that \( \delta \)-near \( s \) the values of \( f(x) \) are above \( f(s) - 1 \). This contradicts \( s_{nk} \) being \( \delta \)-close to \( s \) for large \( k \).

Then, we see that \( m = \inf f(x) \) exists and assuming \( f(x) \) is never equal \( m \), we build \( s_n \) with \( f(s_n) < s + 1/n \). Pick a subsequence converging to \( s \) and see that if \( f(s) > m \) then for \( \varepsilon = (f(s) - m)/2 \) there is a \( \delta \) such that \( \delta \)-close to \( s \) the values of \( f \) are above \( f(s) - \varepsilon = m + \varepsilon \). This contradicts \( s_{nk} \) being \( \delta \) close to \( s \) for large \( k \).

So there is an \( s \in [a, b] \) with \( f(s) \) minimal.

\( \Box \)

**Lemma 0.2.** \( \delta_x(x_0) \) is lower semi continuous.

**Proof.** Let \( e > 0 \). We want to find \( d \) such that any \( x \) that is \( d \)-close to \( x_0 \) has \( \delta(x) \) at least \( \delta_x(x_0) - e \).

To that effect, consider \( f \) on \( \delta_x(x_0) - \varepsilon/2 \) closed neighbourhood of \( x_0 \). This is smaller than \( \delta_x(x_0) \) so the function \( f \) stays within \( \varepsilon \) of \( f(x_0) \). As \( f \) is continuous, the values of \( f \) on this closed segment are a segment \([m, M]\) strictly contained in \([f(x_0) - \varepsilon, f(x_0) + \varepsilon]\). We will use the gap \( g = \min(m - (f(x_0) - \varepsilon), (f(x_0) + \varepsilon) - M) \), so that the values in \( \delta_x(x_0) - \varepsilon/2 \) closed neighbourhood of \( x_0 \) are within \( \varepsilon - g \) of \( f(x_0) \).

Since \( f \) is continuous at \( x_0 \) there is a \( \Delta \) such that for all \( x \) that are \( \Delta \)-close to \( x \), the value \( f(x) \) is within \( g \) of \( f(x_0) \). We can take \( \Delta < e/2 \) Then the values in the \( \delta_x(x_0) - \varepsilon/2 - \Delta \) neighbourhood of \( x \) are within \( g + (\varepsilon - g) = \varepsilon \) of \( f(x) \), and hence so are the values in the smaller \( \delta_x(x_0) - \varepsilon \) neighbourhood. So \( \delta_x(x) \) is at least \( \delta_x(x_0) - e \), as wanted.

\( \Box \)

Hence we have minimal \( \delta \) and \( f \) is uniformly continuous as we wished.

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