1 Real numbers.

While rational numbers, being ratios of integers, are fairly straightforward (the fact that may have been slightly obscured by our abstract treatment of them), the real numbers are a bit more mysterious. We are used to their decimal representations, but the question of whether \(1 = 0.999\ldots\) has produced thousands of web pages. We think that they fill up the coordinate line, but may wonder how to express this mathematically. Many have heard that there are ”more real numbers than rational numbers”, but Georg Cantor, who proved this was also one of the first to define the real numbers rigorously - in 1872, a late date indicative of how elusive this definition was for a long time. In our study we may hope to benefit from the intervening 140 years, during which the concept of real number has been clarified quite a bit.

1.1 The basic intuition.

We think of real numbers as representing points on a coordinate line.

- Rational numbers are there, but they miss some points.
- In fact, they are simultaneously dense, and missing points everywhere.
- We can approximate any irrational number by rational numbers,
- So we can think of real numbers as filling in the gaps left by the rationals.

There are several ways to make this precise, but we will not give a rigorous construction of the reals (paragraph 6 in Ross gives an idea of how such things are done). Instead, we shall take a version of “no gaps” statement as an axiom, but first we must prepare.

We will assume that real numbers satisfy the ordered field axioms (that is, we can add, multiply, subtract and divide them in the usual way, as well as compare them to each other and these operations are compatible). We will appeal to our
intuition about the number line to justify one additional axiom - axiom of comple-

teness. And finally we will deduce everything else from the ordered field and comple-

teness axioms.

1.2 Maximum, minimum. Upper bound, lower bound.

Definition 1.1. Let $S$ be a nonempty subset of $\mathbb{R}$. If $S$ contains a largest element $s_0$, that is $s_0$ is in $S$ and $s < s_0$ for all other $s$ in $S$, then we call $s_0$ the maximum of $S$. If $S$ contains a smallest element $s_0$, that is $s_0$ is in $S$ and $s_0 < s$ for all other $s$ in $S$, then we call $s_0$ the minimum of $S$.

Example 1.2. If $S = \mathbb{N} = \{1, 2, \ldots\} \subset \mathbb{R}$ is the set of natural numbers, then $S$ has no maximum, but it does have a minimum, namely 1.

If $S$ is any non-empty subset of $\mathbb{N}$. Then the well-ordering principle says precisely that $S$ has a minimum.

Example 1.3. We have four types of intervals

\[
[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}
\]

\[
(a, b] = \{x \in \mathbb{R} | a < x \leq b\}
\]

\[
[a, b) = \{x \in \mathbb{R} | a \leq x < b\}
\]

\[
(a, b) = \{x \in \mathbb{R} | a < x < b\}
\]

The ones with “$b]$” have a maximum $b$ and the ones with “$b)$” do not have any maximum. This is intuitively obvious, but the proof is not necessarily immediate. Here is one way to prove this, for $(a, b)$: If $x$ is in $(a, b)$ we want to find a number $y$ between $x$ and $b$. One way to think about it is that we should move a small bit off $x$ (to $x + \varepsilon$, as it were) and stay below $b$. This is true, but we do not have a good way to prove this “from general principles” yet - after all, why can we find $\varepsilon$ small enough? Such a principle will be later provided by the Archimedian property, a consequence of the completeness axiom. But for now we can work around this by explicitly taking $\varepsilon$ that will take us half way to $b$ - that is $\varepsilon = \frac{b-x}{2}$ so that

---

*We do not have to assume that real numbers contain the rational numbers - that comes for free in any ordered field. Indeed, we have 1 in the field, and we can define $2 = 1 + 1$ and $3 = 1 + 2$ and so on, as well as $-2 = -1 + (-1)$ and $-3 = -2 + (-1)$ (or by $-n = (-1)n$ - we can prove this is the same). Because $0 < 1$ we get also $1 < 2$ and $2 < 3$ and so on, as well as $\ldots, -3 < -2 < -1 < 0$, so all the integers are different. We can then define $1/n$ as the multiplicative inverse of $n$ for all non-zero integers $n$, and $m/n$ as the product of $m$ and $1/n$. A little more playing with ordered field axioms would prove that all rational numbers constructed this way inside our ordered field are distinct, so we have all of $\mathbb{Q}$ sitting inside as a subset.
\[ y = x + \varepsilon = x + \frac{b-x}{2} = \frac{x+b}{2} \text{ is less than } b \text{ (and larger than } a), \text{ and so is in } S, \text{ and bigger than } x. \text{ So } x \text{ is not a maximum of } S \text{ and } S \text{ has no maximum.} \]

Similarly, the intervals with \( \langle a \rangle \) have minimum \( a \) and the ones with \( \langle a, b \rangle \) have no minimum.

**Example 1.4.** Let \( S = \{ r \mid r \text{ is rational, } 0 \leq r, \text{ and } r^2 \leq 2 \} \), that is \( S \) is the intersection of \( \mathbb{Q} \) with \([0, \sqrt{2}]\). The maximum of \([0, \sqrt{2}]\) is \( \sqrt{2} \), but it is not rational. This suggests that \( S \) has not maximum. To prove this formally we would have to show that for any \( r \in S \) there is a bigger \( t \in S \), that is no matter what rational number \( r \) below \( \sqrt{2} \) we take there is a larger rational \( t \) that is closer to \( \sqrt{2} \). Again, we will be able to do this from general principles by the end of this week, but for now we can only try to be clever. Our old trick of going halfway does not work as \( \frac{r+\sqrt{2}}{2} \) is not rational. Perhaps a new trick can be found, but instead we are going to come back to this example in the next lecture, armed with better tools.

**Example 1.5.** Let \( S = \{ 1, \frac{1}{2}, \frac{1}{3}, \ldots \} = \{ \frac{1}{n} \mid n \in \mathbb{N} \} \). This has a maximum of \( 1 \) and no minimum (this time there is no trick - if \( \frac{1}{k} \) was the minimum it would have to be smaller than \( \frac{1}{k+1} \), which is not true).

**Example 1.6.** Both \( S = \mathbb{Z} \) and \( S = \mathbb{Q} \) have no minima or maxima.

We see that there are two reasons subsets may not have a maximum.

(i) They escape to plus infinity.

OR

(ii) Something happens “locally”, like a missing point \( (b \text{ for } \langle a, b \rangle \text{ or } \sqrt{2} \text{ in Example 1.4).} \)

We shall deal with the first issue first.

---

\(^{1}\) Indeed, for this particular case there is such a trick. Namely, we want, given rational \( x_0 < \sqrt{2} \) to find rational \( x_1 \) with \( x_0 < x_1 < \sqrt{2} \), that is a better approximation of \( \sqrt{2} \). If you think about what you know about approximating numbers and are lucky to have taken a relevant course, you may think ["Newton’s method"]. Specifically, if we are looking for approximation for the root of \( x^2 - 2 \) starting from \( x_0 \), the next approximation in the Newton’s method is \( x_1 = x_0 - \frac{x_0^2 - 2}{2x_0} \), which is actually bigger than \( \sqrt{2} \) and so doesn’t work for approximations from below. But it does work for approximations from above! If \( x_0 > \sqrt{2} \) then we can check easily with algebra that \( x_1 < x_0 \) and \( x_1^2 < \sqrt{2}! \) (You can also see it by drawing the relevant picture of Newton approximation. Note that if \( x_0 \) is rational then so is \( x_1 \).) To get the corresponding result for approximating from below you need to apply Newton approximation to a function \( f(x) \) with a root at \( \sqrt{2} \) but with \( f'(x) \) and \( f''(x) \) of different sign, such as \( f(x) = (x^2 - 2)(x^2 - 4) \). This will work, but the amount of algebra is unpleasant. And what would we do if instead of \( \sqrt{2} \) we had \( \sqrt{3} + \sqrt{7} \)? Or worse yet, \( \pi \)? Cleverness is nice, but sometimes general methods just win.
Definition 1.7. A subset $S$ of $\mathbb{R}$ is called bounded above if there is $M \in \mathbb{R}$ s.t. $s \leq M$ for all $s \in S$. Then $M$ is called an upper bound.

Similarly, a subset $S$ of $\mathbb{R}$ is called bounded below if there is $m \in \mathbb{R}$ s.t. $s \geq M$ for all $s \in S$, and $m$ is called a lower bound.

If $S$ is bounded below and above then we say $S$ is bounded.

You can think of elements (points) of $S$ as cats trying to escape the real number line either to positive infinity to the right, or to the negative infinity to the left. Then having an upper bound is like having a wall that prevents the escape to the right, and having a lower bound prevents the escape to the left. If you have both, the cats are captured and the set $S$ is bounded.

Note that we have $\leq$ and $\geq$ in the definition, so the wall can be put at the same place as the rightmost or leftmost cat. Ouch!

Example 1.8. If $S = \mathbb{N}$ then it is not bounded above, but is bounded below.

Example 1.9. If $S$ is any non-empty subset of $\mathbb{N}$, then the minimum of $S$ is a lower bound, and any number smaller than the minimum is also a lower bound.

For the intervals from $a$ to be $b$ (of any of the four types), $b$ is an upper bound, and any number bigger is an upper bound, while $a$ and any smaller number is a lower bound.

Example 1.10. In general, if $S$ has a maximum, then that is also an upper bound, and if $S$ has a minimum that is also a lower bound.

Also note that if $M$ is an upper bound, any bugger number is also an upper bound, and if $m$ is a lower bound any smaller number is also a lower bound - we can always make the “cat cage” bigger.

Example 1.11. Let $S = \{r | r$ is rational, $0 \leq r,$ and $r^2 \leq 2\}$. Then $\sqrt{2}$ and any larger number is the upper bound, while anything less than $\sqrt{2}$ is not an upper bound.

We ask whether we can make the cage smaller - in particular, whether there is definite smallest size for the cage. We see from the last example that if we could only put the wall at rational points, we would not be able to put it at $\sqrt{2}$, so we would always be able to move it a bit closer; there would not be any smallest upper bound for $S$.

1.3 Supremum, infimum. The completeness axiom.

Definition 1.12. If $S$ is a nonempty bounded from above subset of $\mathbb{R}$, such that the set of upper bounds of $S$ has a minimum $M$, then we say that $M$ is the least upper bound or supremum of $S$. 
Similarly, if $S$ is bounded from below and the set of lower bounds of $S$ has a maximum $m$, then we say that $m$ is the greatest lower bound or infimum of $S$.

We see that when working with examples of bounded above subsets of rational numbers we can’t always find a rational least upper bound, but for bounded subsets of the reals there is always a supremum. This is a key difference.

**Property 1.13.** A bounded above nonempty subset of $\mathbb{R}$ has a least upper bound

This is the axiom of completeness, which differentiates $\mathbb{R}$ from $\mathbb{Q}$. It will give us the Archimedean property and many other useful consequences which we shall explore in the next lecture.