1 Properties of continuous functions III

Last time we saw that the fact that \( f(x) = x^m \) is continuous and monotone for \( x \geq 0 \) allowed us to use IVT construct the inverse \( f^{-1}(y) = y^{\frac{1}{m}} \).

Similarly, for any monotone continuous function \( f: [a, b] \mapsto [f(a), f(b)] \) there exists an inverse \( f^{-1}: [f(a), f(b)] \to [a, b] \). It is easy to see that this inverse is monotone (check this). We may wonder if this inverse is also continuous. To that effect we have the following continuity criterion for monotone functions.

**Proposition 1.1.** If \( f \) is a monotone function \( f: [a, b] \to \mathbb{R} \) then \( f \) is continuous if and only if the set of values of \( f \) is all of the segment \( [f(a), f(b)] \).

**Proof.** If \( f \) is continuous then we know its set of values is in fact the segment from its minimum to its maximum (as consequence of IVT), which for monotone function is exactly \( [f(a), f(b)] \).

The converse is a bit more work. We assume that \( f \) takes all values between \( [f(a), f(b)] \). We use the \( \varepsilon - \delta \) definition of continuity to show that \( f \) is continuous at any point \( x_0 \in (a, b) \) (the case of \( x_0 = a \) or \( x_0 = b \) is left as exercise). To see that \( f \) is continuous at \( x_0 \) we need to make sure that \( f(x) \) is between \( f(x_0) - \varepsilon \) and \( f(x) + \varepsilon \) for \( x \) close to \( x_0 \) (for all \( \varepsilon \) small). Notice that since \( f \) is increasing, the value \( f(x) \) are between \( f(x_l) \) and \( f(x_r) \) precisely when \( x_l < x < x_r \).

This motivates finding \( x_l \) and \( x_r \) such that \( f(x_l) = f(x_0) - \varepsilon \) and \( f(x_r) = f(x_0) + \varepsilon \).

But for small \( \varepsilon \) we have \( f(a) < f(x_0) - \varepsilon < f(x_0) + \varepsilon < f(b) \), so there are such \( x_l \) and \( x_r \). Then \( \delta = \min(x_r - x_0, x_0 - x_l) \) is as wanted in the definition of continuity.

**Exercise 1.2.** Make the necessary changes in the above proof to show that \( f \) is continuous at \( a \) and \( b \).

Applying this criterion to an inverse of a continuous monotone function immediately shows that it is continuous (check this!).
2 Uniform continuity

2.1 A motivation from integration.

In the course of defining Riemann integral, one encounters the following situation. It is well-known that the idea of the integral is to measure the area under the graph of the function \( f \), and making this idea precise is the content of the Riemann integration theory. As a start, one may want to approximate this area by “discretizing” the function.

Namely, suppose \( f : [0, 1] \to \mathbb{R} \) is the function in question (for this discussion we shall assume for convenience that \( f \) is continuous.) Then to consider \( n \)’th approximation we can divide the domain of \( f \) into \( n \) equal segments \( I_{i,n} = [\frac{i}{n}, \frac{i+1}{n}] \) and see that on each such segment the function is between \( M_{i,n} = \max\{f(x) | x \in I_{i,n}\} = f(u_{i,n}) \) and \( m_{i,n} = \min\{f(x) | x \in I_{i,n}\} = f(l_{i,n}) \) (draw a picture).

The area should surely be smaller than the sum of the “upper” areas \( U_n = \sum \frac{1}{n} M_{i,n} \) and bigger than the sum of the “lower” areas \( L_n = \sum \frac{1}{n} m_{i,n} \). If as \( n \) grows the two approximations both converged to the same value (which we could then call the area), we would have \( D_n = U_n - L_n \) for all large \( n \).

This “error” \( D_n \) is the average of the errors \( d_{i,n} = M_{i,n} - m_{i,n} \), and in particular it would be small if each error \( d_{i,n} \) were small. Of course if \( f \) is continuous, as \( n \) grows the error on each segment should get small. For example, on the very first segment we have \( M_{0,n} = f(u_{0,n}) \) and \( m_{0,n} = f(l_{0,n}) \) both within \( \varepsilon \) of \( f(0) \) for \( \frac{1}{n} < \delta \), so \( d_{0,n} = M_{0,n} - m_{0,n} < 2 \varepsilon \), and does gets arbitrarily small as \( n \) grows.

The question is how to make all the errors \( d_{i,n} \) to be small, for all \( i \), as \( n \) grows.

2.2 Uniform continuity

We have the following definition.

**Definition 2.1.** A function \( f \) is called uniformly continuous on its domain, if for every \( \varepsilon \) there exists \( \delta \) such that if \( |x_1 - x_2| < \delta \) we get \( |f(x_1) - f(x_2)| < \varepsilon \).

One way to understand this definition is to compare it with the definition of continuity for \( f \). That says that for any \( \varepsilon \) and any \( x \) there is some \( \delta \). uniform continuity says that there is a \( \delta \) that works for all \( x \) simultaneously, or uniformly, hence the name. (in particular, it should be obvious to you that any uniformly continuous function is continuous. If it is not, try to write a proof.)

To illustrate the difference between the two concepts we shall consider the following example.

**Example 2.2.** Consider \( f(x) = \frac{1}{x} \) on \((0, \infty)\). We know that it is continuous from the limit theorem we proved. Let’s see how we would prove it directly.
For any $x_0$ and $\varepsilon$ we would look for a $\delta$ such that $|f(x_0) - f(x)| = \frac{|x-x_0|}{xx_0}$ is less than $\varepsilon$ whenever $|x - x_0|$ is less than $\delta$. The ratio of the two is $xx_0$ which is close to $x_0$ when $x$ is close to $x_0$. We may conclude $\delta$ that works is on the order of $xx_0^2$.

In fact since $\frac{1}{x}$ is monotone, we know precisely how small $\delta$ has to be - in terms of the previous section $x_l = \frac{1}{x_0 + \varepsilon}$ and $x_r = \frac{1}{x_0 + \varepsilon}$ and $\delta = \frac{xx_0^2}{1+xx_0}$, which is close to $xx_0^2$.

This shows that $f$ is continuous ($\delta$ exists for all $x_0, \varepsilon$) and also that not uniform $\delta$ can exist for all $x_0$, since for a fixed $\varepsilon$ the relevant $\delta$ goes to 0 as $x_0$ goes to 0, as is also clear from the graph of $f$.

On the other hand if we restrict ourselves to functions defined closed intervals, this phenomenon does not occur.

**Proposition 2.3.** If $f : [a, b] \to \mathbb{R}$ is continuous, then it is also uniformly continuous.

**Remark.** Going back to the “integration” example, note that if $f$ is uniformly continuous, then for a given $\varepsilon$ there is $\delta$ and for $n$ with $\frac{1}{n} < \delta$ we get $|u_{i,n} - \lim_{n} u_{i,n} | < \varepsilon$ so $M_{i,n} - m_{i,n} < \varepsilon$, so $U_n - L_n < \varepsilon$, as we wanted.

There is a very natural approach to the above Proposition - simply take the “minimal” of the $\delta$’s that work for each $x$ and set that as the “uniform $\delta$”. Unfortunately, this is quite subtle to implement. In particular, for a fixed $\varepsilon$, the biggest size of $\delta$ that works for $x$ is not continuously dependent on $x$! This makes finding minimal $\delta$ rather tricky. This can be made to work using the notion of lower semi-continuity and is the subject of Supplement 3.

Next time we will give an more indirect but rather simpler proof of this proposition.