1 Continuity.

1.1 Continuous functions.

When we talked about sequences we defined them as functions from \( \mathbb{N} \) to \( \mathbb{R} \). We noted then that a function is not the same thing as a formula. The important thing was that for every input \( n \) we get an output \( s_n \). Similarly, functions \( f : \mathbb{R} \to \mathbb{R} \) do not have to be given by a formula. What matters is that to any input \( x \) the function \( f \) associates a unique output. Sometimes we will deal with functions defined on a subset of \( \mathbb{R} \). In general, a real valued function takes elements from some subset \( S \) of \( \mathbb{R} \) to values in \( \mathbb{R} \). Then \( S \) is called the domain of definition of \( f \) and we write \( f : S \to \mathbb{R} \).

We now focus on continuous functions. One view of continuity is the absence of discontinuity - jumps and other “bad behaviour”. That is, the graph of a continuous function on \( \mathbb{R} \) can be traced out “without lifting chalk from the blackboard”. Translating this intuition into mathematics is not quite obvious, but the idea is that the nearby points in the domain should go to nearby points of the range. The key point is that for any size of the “output box” around \( f(x) \) we should be able to find small enough “input box” around \( x \) that all inputs in the input box produce output in the output box. Written formally, this gives the following definition.

**Definition 1.1** (Definition 1). A real valued function \( f \) is continuous at point \( x \) of its domain if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( |x - x'| < \delta \) then \( |f(x) - f(x')| < \varepsilon \). We say that \( f \) is continuous if \( f \) is continuous at all points of the domain.

That is, we can make the change in \( f \) arbitrarily small if we stay close to \( x \).

*Some more formally-minded of you may think that “a rule assigning outputs to inputs” is kind of a vague definition. In fact, there is a rigorous definition which basically defines \( f \) as its graph!"
1.2 An application to limits.

Sometimes when trying to prove that a sequence \( s_n \) converges to \( s \) one is able to show that for any \( \varepsilon > 0 \) there is an \( N \) such that for \( n > N \) we have \( |s_n - s| < f(\varepsilon) \). That is, \( |s_n - s| < \varepsilon^2 \) or \( |s_n - s| < \varepsilon + \varepsilon^3 \) or some such. Is this enough to conclude that \( s_n \) converges to \( s \)?

Well, if for any \( \varepsilon \) we can get \( f(\varepsilon) < \varepsilon \) for some \( \varepsilon > 0 \) then we know that for all \( n > N(\varepsilon) \) we have \( |s_n - s| < f(\varepsilon) < \varepsilon \) and we are done.

Note that if \( f(0) > 0 \) then we likely don’t get anything, as, for example \( |s_n - s| < 2 + \varepsilon^2 \) will not force \( |s_n - s| \) to be small.

On the other hand if \( f(0) = 0 \), then \( f(\varepsilon) < \varepsilon \) for small \( \varepsilon \) would follow from continuity of \( f \) at 0. And indeed, all in most cases we are likely to run into such a thing we will have \( f(0) = 0 \) and \( f \) continuous, so we can indeed conclude that \( \lim s_n = s \).

1.3 Continuous functions, continued.

There is an alternative perspective on continuity that says that continuity is the property of taking limits to limits.

**Definition 1.2** (Definition 2). A real valued function \( f \) is continuous at point \( x \) of its domain if for any sequence \( (x_1, x_2, \ldots) \) of points in the domain converging to \( x \) the sequence of values \( f(x_1), f(x_2), \ldots \) converges to \( f(x) \). We say that \( f \) is continuous if \( f \) is continuous at all points of the domain.

Note that from this perspective the limit theorem saying that \( \frac{1}{x} \) converges to \( \frac{1}{x} \) if \( s_n > 0 \) converges to \( s > 0 \) is saying precisely that \( f(x) = \frac{1}{x} \) is continuous on the domain \( x > 0 \). Similarly \( \sqrt{s_n} \) converges to \( \sqrt{s} \) if if \( s_n > 0 \) converges to \( s > 0 \) is the continuity of \( \sqrt{x} \) on domain \( x > 0 \).

We now have two definitions for continuity at a point \( x \). Not to worry.

**Theorem 1.3.** These two definitions are equivalent.

**Proof.** 1 implies 2. Let \( x_n \) be converging to \( x \). We aim to show \( f(x_n) \) converge to \( f(x) \), that is for any \( \varepsilon > 0 \) we want \( |f(x) - f(x_n)| < \varepsilon \). Since \( f \) is continuous in the sense of definition 1 this is ensured by \( |x_n - x| < \delta \). But since \( x_n \) converge to \( x \), there is \( N \) such that for \( n > N \) we will indeed get \( |x_n - x| < \delta \). We are done.

2 implies 1. We will prove the contrapositive – not continuous by Definition 1 implies not continuous by Definition 2. Namely, we suppose there is \( \varepsilon > 0 \) such that for any \( \delta > 0 \) there is some \( x_\delta \) which is \( \delta \) close to \( x \) but \( |f(x_\delta) - f(x)| > \varepsilon \). Take \( \delta \) to be \( 1, \frac{1}{2}, \frac{1}{3}, \ldots \). We get a sequence \( x_1, x_{\frac{1}{2}}, \ldots \) converging to \( x \) but \( f(x_i) \) stays away from \( f(x) \) by at least \( \varepsilon \), so will not converge to \( f(x) \). QED.