Real Analysis Lecture 17

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1 Series

1.1 Definition and basic examples.

We use the standard notation $\sum_{k=m}^{n} a_k$ for the sum $a_m + a_{m+1} + \ldots + a_n$.

The symbol $\sum_{k=m}^{\infty} a_k$ is given meaning by considering the sequence of partial sums $s_n = \sum_{k=m}^{n} a_k$; by definition $\sum_{k=m}^{\infty} a_k = \lim s_n$, if the limit exists. By convention this extends to the cases when $s_n$ diverges, so that $\sum_{k=m}^{\infty} a_k$ can be $\infty$ or $-\infty$.

Example 1.1 (Non-example). Let $a_n = (-1)^{n+1}$. Then $s_1 = a_1 = 1$, $s_2 = 1 - 1 = 0$, $s_3 = 1$ and so on, so that $s_n = 1$ if $n$ is odd, 0 if $n$ is even. Then $\sum_{k=1}^{\infty} a_k$ does not make sense.

If all $a_n \geq 0$ then $s_n$ is monotone non-decreasing, so it either converges or diverges to $+\infty$, and so $\sum_{k=1}^{\infty} a_k$ always makes sense in this case. In particular, $\sum_{k=1}^{\infty} |a_k|$ is always meaningful for any sequence $a_k$.

We note that in order for $\sum a_k$ to converge it is necessary for the terms $a_k$ themselves to go to zero. Indeed if $s_n$ and $s_{n+1}$ are within $\varepsilon$ of the limit $s$, then their difference $s_{n+1} - s_n = a_n$ must be smaller than $2\varepsilon$. This explains what happened in our non-example as well - the $a_n$ were $\pm 1$, and so did not go to zero.

We have the following fundamental examples:

Example 1.2. The geometric sequence $\sum_{k=0}^{\infty} ar^k$ has, for $r \neq 1$, partial sums $\sum_{k=0}^{n} ar^k = a \frac{1-r^{n+1}}{1-r}$ so converges to $\frac{a}{1-r}$ when $|r| < 1$. It diverges when $r \geq 1$ and does not have a limit if $r \leq -1$.

Example 1.3. The $p$-series $\sum_{n=0}^{\infty} \frac{1}{n^p}$ for a fixed positive $p$. For $p = 1$ you get the harmonic series $\sum_{n=0}^{\infty} \frac{1}{n}$, which you will prove divergent in homework. Since for $\sum_{n=0}^{\infty} \frac{1}{n^p} > \sum_{n=0}^{\infty} \frac{1}{n}$ for $p < 1$ we see that the $p$-series with $p < 1$ diverges. For any $p > 1$ the $p$-series converges, which we shall prove when we talk about the integral test. See also this note for an alternative proof.
This shows that it is not enough for \( a_n \) to go to zero, they have to do so sufficiently fast. Determining how fast is fast enough is precisely the content of various convergence criterions, which we shall treat shortly, right after we describe the Cauchy criterion of convergence.

### 1.2 Cauchy criterion and alternating series.

The question of series convergence is really a special case of the question of sequence convergence. We had an alternative definition of convergence via Cauchy sequences. Let’s see what it gives for series.

The sequence of partial sums \( s_n \) is Cauchy if for any \( \varepsilon > 0 \) there is an \( N \) such that for all \( n, m > N \) we have \( |s_n - s_m| < \varepsilon \). We may as well assume \( n > m \) (why?). But then \( s_n - s_m = a_{m+1} + \ldots + a_{n-1} \). So the series converges if and only if for any \( \varepsilon > 0 \) exists \( N \) such that for \( n \geq m > N \) we get \( \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon \).

There is another general situation where convergence is automatic.

**Proposition.** (Alternating series converge). If \( a_n \) is non-increasing sequence of positive terms converging to zero, then \( \sum (-1)^n a_n \) converges.

The point is that partial sums \( s_n = \sum_{k=1}^{n} (-1)^n a_n \) for \( n > N \) stay between \( s_N \) and \( s_{N+1} \), and hence their sequence is Cauchy (since the distance \( |s_{N+1} - s_n| = a_{N+1} \) goes to zero.).

**Proof.** Claim: \( s_{2l-1} \leq s_n \leq s_{2l} \) for all \( n > 2l \). Proof: \( s_{n+1} - s_{2l-1} = (a_{2l} - a_{2l+1}) + (a_{2l+2} - a_{2l+3}) + \ldots + (a_{n-1} - a_n) \) for odd \( n \) and \( s_{n+1} - s_{2l-1} = (a_{2l} - a_{2l+1}) + (a_{2l+2} - a_{2l+3}) + \ldots + (a_{n-2} - a_{n-1}) + a_n \) for even \( n \). Since in either case all terms are non-negative, the inequality follows.

Similarly \( s_n - s_{2l} = (-a_{2l+1} + a_{2l+2}) + (-a_{2l+3} + a_{2l+4}) + \ldots + (-a_{n-1} + a_n) \) if \( n \) is even, or \( s_n - s_{2l} = (-a_{2l+1} + a_{2l+2}) + (-a_{2l+3} + a_{2l+4}) + \ldots + (-a_{n-2} + a_{n-1}) - a_n \) if \( n \) is odd. In any case, all the terms are non-positive, so the other inequality follows.

This proves the claim.

Hence \( s_n \) is Cauchy and we are done. \( \square \)