1 Convergent subsequences.

Last time we saw that all positive numbers in $\mathbb{Q}$ can be listed as a sequence.

**Proposition.** This sequence has a subsequence converging to any positive real number $x$.

**Proof.** We will make a monotone non-decreasing sequence convergent to $x$ by constructing it one by one.

First we use the density of $\mathbb{Q}$ to pick rational $0 < t_1 < x$. Then $t_1 = s_{n_1}$ for some index $n_1$. We want to pick rational $t_2$ closer to $x$. We could just take $t_1 < t_2 < x$, but this is too weak. We will take $t_2$ at least twice as close to $x$ as $t_1$, that is $x - t_2 < \frac{x-t_1}{2}$ aka $\frac{x+t_1}{2} < t_2 < x$. There is certainly such a rational $t_2$, and $t_2 = s_{n_2}$. The problem is that we want to have a subsequence of $s_n$, not just a subset of values, that is we want $n_2 > n_1$. That’s not a problem though - there are infinitely many rational numbers between $\frac{x+t_1}{2}$ and $x$ and only finitely many of them could possibly be one of $s_1, s_2, \ldots, s_{n_1}$, so we can pick one that isn’t - that one will be $s_{n_2}$ for some $n_2 > n_1$. All this work to build $t_2$. But now we are almost done. By the same argument we can find a rational $t_3 = s_{n_3}$ with $n_3 > n_2$ such that $\frac{x+t_2}{2} < t_3 < x$, then $t_4 = s_{n_3}$ and so on by induction. The resulting sequence is monotone increasing subsequence of $s_n$ and we know $x - t_n < \frac{x-t_1}{2^n}$ which is less than $\varepsilon$ for large $n$ (this is why we wanted to half the distance at each step). This means $t_n$ converges to $x$, as wanted.

Exercise: Prove that $s_n$ has a decreasing subsequence converging to 0.

Exercise: Prove that $s_n$ has a subsequence diverging to $+\infty$.

We have the fundamental theorem.

**Theorem 1.1.** Every bounded sequence has a subsequence which is Cauchy.
The idea is to force elements \( t_k \) of a subsequence into a smaller and smaller box \( B_k \) as \( k \) gets larger. It is good to draw a picture of these boxes as you follow the proof below.

**Proof.** We have \( m = m_1 < s_n < M = M_1 \), and \( B_1 = [m_1, M_1] \) be the first box. Let \( n_1 = 1 \) so that \( t_1 = s_1 \in B_1 \).

Then let \( a_1 = \frac{m_1 + M_1}{2} \), so \( a_1 \) divides the box \( B_1 \) into the left box \( L_1 = [m_1, a_1] \) and right box \( R_1 = [a_1, M_1] \).

We look at the set \( S_1 = \{ s_n \} \).

Then as \( S_1 \cap B_1 \) is infinite, at least one of \( S_1 \cap L_1 \) and \( S_1 \cap R_1 \) is infinite. In the first case denote \( m_2 = m_1 \) and \( M_2 = a_1 \) and in the second case denote \( m_2 = a_1 \) and \( M_2 = M_1 \) (if both are infinite do either). The second box is \( B_2 = [m_2, M_2] \).

Note that the length of \( [m_2, M_2] \) is half that of \( [m_1, M_1] \). Let \( n_2 \) be the such that \( n_2 > n_1 \) and \( t_2 = s_{n_2} \in B_2 \) and denote \( S_2 = S_1 \cap B_2 \).

We now repeat this process.

Let \( a_2 = \frac{m_2 + M_2}{2} \), so \( a_2 \) divides the box \( B_2 \) into the left box \( L_2 = [m_2, a_2] \) and right box \( R_2 = [a_2, M_2] \).

Then as \( S_2 \cap B_2 = S_2 \) is infinite, at least one of \( S_2 \cap L_2 \) and \( S_2 \cap R_2 \) is infinite. In the first case denote \( m_3 = m_2 \) and \( M_3 = a_2 \) and in the second case denote \( m_3 = a_2 \) and \( M_3 = M_2 \) (if both are infinite do either). Note that the length of \( [m_3, M_3] \) is half that of \( [m_2, M_2] \). Let \( a_3 = \frac{m_3 + M_3}{2} \). Let \( n_3 \) be the such that \( n_3 > n_2 \) and \( s_{n_3} \in [m_3, M_3] \) and denote \( S_3 = S_2 \cap [m_2, M_2] \).

We proceed in this manner building \( n_k \) (and \( m_k, M_k, a_k, B_k, L_k, R_k \) and \( S_k \)) by induction as we go along. We note that \( s_{n_k} \in S_K \) for all \( k > K \) and \( S_K \subset B_K = [m_K, M_K] \) so \( |s_{n_k} - s_{n_{k+1}}| < M_K - m_K = \frac{M_1 - m_1}{2^{k+1}} < \varepsilon \) for large enough \( K \). So indeed \( s_{n_k} \) is a Cauchy subsequence of \( s_k \).

\( \Box \)

**Corollary 1.2** (Bolzano-Weierstrass). Every bounded sequence of real numbers has a convergent subsequence.

### 1.1 Monotone subsequence.

It turns out we can do better.

Namely, we can now prove the following stronger theorem.

**Theorem 1.3.** Every sequence \( s_n \) for which \( s = \limsup s_n \) is finite contains a monotone subsequence converging to \( s \).

Of course this applies to bounded sequences, and so reproves Bolzano-Weierstrass theorem above. This proof is, however, more complicted. Bolzano-Weierstrass is a fundamental result, so I felt it would be worthwhile to see a more direct proof.
before embarking to prove the stronger version presented here. Apart from it’s intrinsic interest, this stronger version lets us characterize the set of all subsequential limits of $s_n$ better, and will also be used later when we discuss power series.

We will prove this next time.