1 \(\liminf\), \(\limsup\) and \textbf{Cauchy sequences.}

Last time we proved

**Proposition.** If \(\limsup s_n = \liminf s_n = s\) then \(s_n\) converges to \(s\).

In this section we will apply the properties of \(\liminf\) and \(\limsup\) to get a new understanding of convergence.

We said that a sequence converged if it got close to a fixed number \(s\), the limit. It turns out that for real numbers that is equivalent to the elements of the sequence getting close to one another.

More precisely we have the following definition.

**Definition 1.1.** A sequence \(s_n\) is called Cauchy if for any \(\varepsilon > 0\) there is \(N\) such that for all \(n, m > N\) we get \(|s_n - s_m| < \varepsilon\).

Note that this definition has very similar structure to the definition of convergent sequence - for every gauge of smallness \(\varepsilon\) there is a point \(N\) after which some closeness property holds. The only (crucial) difference is that closeness is to each other, \(|s_n - s_m| < \varepsilon\), rather than to a predetermined number.

One may suspect that being close to a fixed number \(s\) would make terms close to each other as well, that is that all convergent sequences are Cauchy. One would be right.

**Proposition 1.2.** If \(s_n\) converges to \(s\), then \(s_n\) is Cauchy.

**Proof.** For any \(\varepsilon > 0\) we have for all \(n > N\) that \(|s_n - s| > \varepsilon/2\), and for all \(m > M\) also \(|s_m - s| < \varepsilon/2\). Hence by triangle inequality \(|s_n - s_m| \leq |s_n - s| + |s - s_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon\), and so \(s_n\) is Cauchy. \(\square\)
The main point of this section is that, for real numbers, the converse also holds. We should note that this is more subtle, as it fails for $\mathbb{Q}$ (note that the previous proposition used only the definitions of convergent and Cauchy and triangle inequality, so it held for $\mathbb{Q}$ as well). For example $S = \{ r | r^2 < 2 \}$ has $\sup S = \sqrt{2}$ and we can build a sequence of rational numbers increasing to $\sqrt{2}$ (we shall show this rigorously in next lecture). Such a sequence will converge in $\mathbb{R}$, and in particular will be Cauchy, but will fail to converge in $\mathbb{Q}$, as the number to which it is supposed to converge is a hole in $\mathbb{Q}$. The fact that this does not happen in $\mathbb{R}$ is a manifestation of completeness of $\mathbb{R}$, and so must use the completeness axiom. Our proof will rely on the completeness axiom through its use of $\lim \inf$ and $\lim \sup$ - the fact that these exist relies on completeness (through the theorem that monotone increasing sequences converges to it’s supremum, which in turn requires that the supremum actually exists - by completeness axiom of $\mathbb{R}$).

With this introduction, we can now prove the result.

**Theorem 1.3.** Every Cauchy sequence of real numbers converges.

**Lemma 1.4.** Cauchy sequences are bounded.

**Proof.** We show the tail is bounded. This is fairly clear: There exists $N$ such that for all $n > N$ we have $|s_n - s_{N+1}| < 1$, so $b = s_{N+1} - 1 < s < s_{N+1} + 1 = B$.

The initial segment is bounded - the set $\{s_1, \ldots, s_N\}$ has a maximum $M$ and a minimum $m$.

Then the whole sequence is bounded by $\min \{b, m\}$ below and by $\max \{B, M\}$ above. QED. 

**Proof.** By lemma, it suffices to show that for a Cauchy sequence $\lim \sup s_n = \lim \inf s_n$.

Just following our nose we write out the hypothesis that $s_n$ is Cauchy: $|s_m - s_n| < \varepsilon$ for all $n > N$, so that $s_m - \varepsilon < s_n < s_m + \varepsilon$.

We want to connect to $\lim \sup s_n$ and $\lim \inf s_n$, so it’s good to notice $s_m + \varepsilon$ is an upper bound for $\{s_n \mid n > N\}$, and so $v_N < s_m + \varepsilon$. Similarly $s_m + \varepsilon$ is a lower bound and $u_N > s_m - \varepsilon$. So $v_N < u_N + 2\varepsilon$.

But $v_N$ is decreasing to $\lim \sup s_n$ and $u_N$ is increasing to $\lim \inf s_n$, so $\lim \sup s_n + 2\varepsilon < v_N + 2\varepsilon < u_N < \lim \inf s_n$. Since this is true for all $\varepsilon$, we get $\lim \sup s_n < \lim \inf s_n$, so they are equal and $s_n$ converges. QED.

Thus a sequence converges if and only if it is Cauchy. The advantage is that we don’t have to specify the limit in advance.
2 Subsequences.

Sometimes when good things like convergence don’t happen to a sequence, they still happen to a subsequence.

Definition 2.1. A sequence $t_k$ is a subsequence of $s_n$ if for each $k$ there is $n_k$ such that $t_k = s_{n_k}$ and $n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots$ (that is $n_k$ is a strictly increasing sequence of integers).

We can specify $t_k$ simply by selecting the $n_k$, or even by specifying an infinite $S$ subset of $\mathbb{N}$ ($S$ will have a smallest element $n_1$, then $S_1 = S \setminus n_1$ will have the smallest element $n_2$ and so on, so we can build $t_k = s_{n_k}$ by induction on $k$).

Examples: Subsequence fo positive terms of $(-1)^n$, subsequence of non-negative terms of $\sin\left(\frac{\pi n}{3}\right)$.

Example: $\mathbb{Q}$ can be listed as a sequence. This sequence has a subsequence converging to any real number.

Next time we will prove the fundamental theorem.

Theorem 2.2. Every bounded sequence has a subsequence which is Cauchy.