1 \( \liminf, \limsup \) and their properties.

A comment on last lecture: Operating with suprema seems to be a challenge. As emphasized in Grader Comments, there are two facts about a supremum:

1) It is an upper bound (so any element is below it) and
2) It is the smallest of the bounds (so any other bound is above it).

We also have the the fact from homework:

3) The set gets arbitrarily close to its supremum (for any \( a \) below supremum, there is an element of the set above it).

Whenever you are faced with a supremum try applying these 2 facts. If needed, apply the third.

This is particularly relevant for lim sup’s where all basic proofs are based on the fact that \( v_N \) are suprema. (In practice, faced with a basic \( \limsup \) problem, try to write it in terms of \( v_N \); more advanced proofs about \( \limsup \) can also be based on Propositions 1 and 2 from last lecture and the statements deduced today). To get a bit more practice we get into the following digression.

1.1 An interlude

Proposition 1.1 (A useful meta observation, version 1). Saying \( (P(s_n) \text{ holds for finitely many } n) \) is equivalent to (there exists \( N \) such that \( P(s_n) \) holds only for \( n < N \)).

Convince yourself that this is true (if needed, draw some dots, one for each natural number, orange if \( P(s_n) \) is true and white if it’s false).

Proposition 1.2 (A useful meta observation, version 2). Saying (for any \( N \) exists \( n > N \) such that \( P(s_n) \) is true) is equivalent to \( (P(s_n) \text{ holds for infinitely many } n) \).
Convince yourself that this is the contrapositive of Version 1, and hence also true.

Given a sequence $s_n$ and a real number $a$ consider the following 4 statements:

(i) $\sup s_n > a$. There exists $n$ such that $s_n > a$.

(ii) $\inf s_n > a$. For all $n$ we have $s_n > a$.

(iii) $\limsup s_n > a$. This means every $v_N > a$, so for any $N$, there is $s_n > a$ with $n > N$, so There are infinitely many $n$ with $s_n > a$.

(iv) $\liminf s_n > a$. This means that $u_N > a$ eventually, so $s_n \geq u_N$ for all $N > n$, that is $s_n < a$ for only finitely many $n$.

In other words, taking $a$ only $\varepsilon$ smaller than the thing it’s compared to:

(i) $s_n$ gets above $\sup s - \varepsilon$ at least once. NB this can be achieved by $s_n = s$ for some $s$, which can happen as infrequently as one time.

(ii) $s_n$ stays above $\inf s - \varepsilon$ always.

(iii) $s_n$ gets above $\limsup s_n - \varepsilon$ infinitely often.

(iv) $s_n$ stays above $\liminf s_n - \varepsilon$ eventually.

Exercise: Replace “$> a$” with “$< a$” and switching $\sup$ and $\inf$ in the first 4 statements, write out corresponding conclusions. Using $a$ just $\varepsilon$ bigger than the thing it’s compared to, deduce:

(i) $s_n$ gets bellow $\inf s + \varepsilon$ at least once.

(ii) $s_n$ stays bellow $\sup s + \varepsilon$ always.

(iii) $s_n$ gets bellow $\liminf s_n + \varepsilon$ infinitely often.

(iv) $s_n$ stays bellow $\limsup s_n + \varepsilon$ eventually.

Putting these 8 statements together we see:

(i) $\sup s_n$ is the divider between those numbers above which $s_n$ goes at least once and those numbers above which it never goes.

(ii) $\limsup s_n$ is the divider between those numbers above which $s_n$ goes infinitely often and those numbers above which it goes only finitely many times.
(iii) $\inf s_n$ is the divider between those numbers below which $s_n$ goes \textit{at least once} and those below which it \textit{never} goes.

(iv) $\liminf s_n$ is the divider between those numbers below which $s_n$ goes \textit{infinitely often} and those numbers below which it goes only \textit{finitely many times}.

Exercise: How do these 4 statements and the ones before them relate to the idea of $\sup$ and $\inf$ giving the best absolute box for $s_n$, and $\limsup$ and $\liminf$ giving the best eventual box for $s_n$?

1.2 $\liminf$, $\limsup$ \textbf{and convergence}.

Getting back to our main line, last time we proved

\textbf{Proposition.} If $\limsup s_n = \liminf s_n = s$ then $s_n$ converges to $s$.

Note that $\limsup s_n = \liminf s_n = \infty$ means $u_N$ diverge to $+\infty$, so for any $M$ there is $N$ with $u_N > M$ and hence $s_n \geq u_N < M$. and $s_n$ diverges to $+\infty$. Similar result holds for $\limsup s_n = \liminf s_n = -\infty$. So the Proposition extends to those cases.

We note that the converse of this also holds.

\textbf{Proposition.} If $s_n$ converges to $s$ then $\limsup s_n = \liminf s_n = s$

\textit{Proof.} If $s_n$ converges to $s$ then for any $\varepsilon$ there are only finitely many $s_n$ above $s + \varepsilon$. Hence by Propositions 1 and 2 from last time, $\limsup s_n < s + \varepsilon$.

Since this is true for any $\varepsilon > 0$, we conclude $\limsup s_n \leq s$.

Similarly, by the infimum versions of Propositions 1 and 2 from last time, $\liminf s_n > s - \varepsilon$ for all $\varepsilon > 0$, so $\liminf s_n \geq s$.

So we have $\liminf s_n \geq s \geq \limsup s_n$, and since $\limsup s_n \geq \liminf s_n$ always, we conclude that they are all equal, as wanted.

We remark (and leave to you to prove) that this extends to the cases $s_n = \pm\infty$ as well.