1 Properties of \( \lim \inf \) and \( \lim \sup \).

Last time for a bounded sequence \( s_n \) we defined \( v_N = \sup\{s_n | n > N\} \) and \( u_N = \inf\{s_n | n > N\} \) and since \( v_N \) and \( u_n \) are bounded monotone, they converge, so we can define \( \lim \sup s_n = \lim v_N \) and \( \lim \inf s_n = \lim u_N \).

Note that \( v_N \geq u_N \), so \( \lim \sup s_n \geq \lim \inf s_n \).

We can extend these definitions to the case when \( s_n \) is not bounded. For example, when \( s_n \) is not bounded above, we have \( v_n = +\infty \) and we put \( \lim \sup s_n = +\infty \). Similarly, for a sequence not bounded below, \( u_N = -\infty \) and we put \( \lim \inf s_n = -\infty \). Not that the converse also holds - if \( \lim \sup s_n = \infty \) then \( s_n \) is not bounded from above, and if \( \lim \inf s_n = -\infty \) then \( s_n \) is not bounded from below.

There is however another issue.

Example 8: \( s_n = n \) This is not bounded above, so \( \lim \sup s_n = \infty \). But note \( u_N = N + 1 \), which also diverges to \( -\infty \). So \( \lim \inf s_n = -\infty \).

Example 9: \( s_n = n \) if \( n \) is odd and \( -\frac{1}{n} \) if \( n \) is even. Then \( \lim \sup s_n = \infty \) and \( \lim \inf s_n = 0 \) (check this!).

So in general \( \lim \sup s_n \) and \( \lim \inf s_n \) always make sense, and \( \lim \sup s_n \geq \lim \inf s_n \) but either or both may be infinite. We shall concentrate on bounded sequences, making only occasional remark about the unbounded ones.

To get a bit more intuition about \( \lim \inf \) and \( \lim \sup \), we have the following propositions:

**Proposition 1.1.** If \( B > s = \lim \sup s_n \) then there exists \( N \) such that \( s_n < B \) for all \( N > n \).

**Proof.** As \( v_N \) are decreasing converging to \( s \) we get \( v_N < B \) for some \( N \)(Details: For large \( N \) we have \( |v_N - s| = v_N - s = \varepsilon < B - s \), so \( v_N < B \)). Then as \( v_N \) is a supremum, \( s_n \leq v_N < B \) for all \( n > N \). \( \square \)
Proposition 1.2. If $B < \limsup s_n$ then for any $N$ there is $n > N$ with $s_n > B$. That is, $s_n > N$ infinitely often.

Proof. As $v_N$ decreases to a value above $B$, we get $v_N > B$ for all $N$. Since $v_N$ is the supremum of $\{s_n | n > N\}$, it is the lowest upper bound, so $B > v_N$ is not an upper bound, which says we can get $s_n$ above $B$ for some $n > N$. This holds for arbitrary $N$ so we are done.

So $\limsup s_n$ is the divider between those $B$ for which $s_n$ is eventually below them and those for which $s_n$ goes above $B$ infinitely often.

Note that for $\limsup$ itself it can go either way - the sequence may never go above it, like for the $s_n = 1 - \frac{1}{n}$, or always go above it like $s_n = \frac{1}{n}$, or alternate as $s_n = (-1)^n \frac{1}{n}$, or equal it all the time like the constant sequence, or equal some of the time and not equal another part - anything goes really.

We have similar propositions for $\liminf s_n$.

Proposition 1.3. If $b < \liminf s_n$ then there exists $N$ such that $s_n > b$ for all $N > n$.

Proposition 1.4. If $b > \liminf s_n$ then there are infinitely many $s_n$ with $s_n < b$.

We have said that $\limsup$ and $\liminf$ determine the size of the box close to which $s_n$ will stay eventually ($s_n$ may venture above $\limsup$ or below $\liminf$ but by smaller and smaller amounts). So if the box is of size zero, then $s_n$ eventually stays close to some particular number, so it stands to reason that that number is $s_n$’s limit.

Proposition. If $\limsup s_n = \liminf s_n = s$ then $s_n$ converges to $s$.

Proof. For a given $\varepsilon > 0$ we need $s - \varepsilon < s_n < s + \varepsilon$ for all $n > N$. This is basically saying that $s + \varepsilon$ is an upper bound for $\{s_n | n > N\}$ and $s - \varepsilon$ is a lower bound. Those are true precisely when $v_N \leq s + \varepsilon$ and $u_N \geq s - \varepsilon$. But both $v_N$ and $u_N$ converge to $s$, so that holds for large $N$.

The formal proof is written in reverse.

For any $\varepsilon$ there exists $N_1$ such that $v_{N_1} < s + \varepsilon$ and exists $N_2$ such that $u_{N_2} > s - \varepsilon$. Then for $N > \max\{N_1, N_2\}$ we get $s + \varepsilon > v_N \geq s_n \geq u_N > s - \varepsilon$, so that $|s - s_n| < \varepsilon$, as wanted.

There is also an alternative proof.

Proof. We have $u_n \leq s_{n+1} \leq v_n$, and as $\lim v_n = \lim s_n = s$ by the squeeze theorem $s_{n+1}$ converges to $s$ as well, but then so does $s_n$. 

2
Note that $\limsup s_n = \liminf s_n = \infty$ means $u_N$ diverge to $+\infty$, so for any $M$ there is $N$ with $u_N > M$ and hence $s_n \geq u_N < M$. and $s_n$ diverges to $+\infty$. Similar result holds for $\limsup s_n = \liminf s_n = -\infty$. So the Proposition extends to those cases.

Next time we will show the converse and use these “limit detection” properties of $\liminf$ and $\limsup$ to get an equivalent definition of convergent sequence without explicitly mentioning the limit!