Having dealt with basic properties of limits, we can now investigate some conditions that will allow us to conclude that a sequence converges without knowing the limit a priori.

1 Monotone sequences.

1.1 Monotone sequences and their convergence.

**Definition 1.1.** A sequence \( s_n \) is nondecreasing if \( s_{n+1} \geq s_n \) for all \( n \). Similarly, a sequence \( s_n \) is nonincreasing if \( s_{n+1} \leq s_n \) for all \( n \).

Note that the sequence that is nondecreasing and nonincreasing is constant. Also note that for non-decreasing sequence \( s_n \geq s_m \) whenever \( n \geq m \).

A sequence that is either non-increasing or non-decreasing is called monotone.

Examples:
- \( 1 - \frac{1}{n^2}, \sqrt{n} \)
- \( (-1)^n, \cos(n) \)
- \( \frac{1}{n^2} \)

(this last one is monotone decreasing for \( n > 3 \), see footnote for this lecture).

**Theorem 1.2.** All bounded above non-decreasing sequences converge to their supremum.

A sequence that is bonded has a supremum. In general it may com by the supremum (or equal it) and then go back down; but a non-decreasing sequence can not go back down, so it stays near the supremum. hence converges to it. Now we write this as a proof.

**Proof.** We treat the non-decreasing case. As \( s_n \) is bounded, it has a supremum \( B \). By homework, for any \( \varepsilon \) there is an element \( s_N \) s.t. \( B - \varepsilon \leq s_N \). Then by monotonicity, \( B - \varepsilon < s_N < s_n \) for all \( n > N \), and since \( B \) is an upper bound \( B - \varepsilon < s_n \leq B \), so \( s_n \) converges to \( B \).
Now, as usual, we can prove that all bounded below non-increasing sequence converge to their infimum. We can mimic the proof above, or we can use that \( \inf S = \sup -S \) and \( \lim -s_n = - \lim s_n \).

It is an easy exercise to show that an unbounded non-decreasing sequence diverges to \(+\infty\), and unbounded non-increasing one diverges to \(-\infty\). So for monotone sequences the expression \( \lim s_n \) always makes sense.

1.2 Decimals.

We will now show that any decimal expansion gives a real number. More precisely, for us a positive decimal expansion is a non-negative integer number \( k \), together with a sequence \( d_i \) with \( d_i \) being a decimal digit, that is a member of the set \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \). Then we have a sequence of partial expansions

\[
s_0 = k, \quad s_1 = k + \frac{d_1}{10}, \quad s_2 = k + \frac{d_1}{10} + \frac{d_2}{100}, \ldots, \quad s_n = k + \frac{d_1}{10} + \frac{d_2}{100} + \ldots + \frac{d_n}{10^n}.
\]

Clearly \( s_n \) is non-decreasing. It is also true that replacing \( d_i \) with 9 gives a sequence \( t_n \) (\( t_0 = k, t_1 = k + \frac{9}{10}, \ldots, t_n = k + \frac{10^n - 1}{10^n} \)) which converges to \( k + 1 \), and such that \( s_n \leq t_n < k + 1 \), so \( s_n \) is bounded, hence convergent. The limit of \( s_n \) is what is meant by the real number represented by \( k.d_1d_2d_3\ldots \). In particular, \( 0.999\ldots = \lim \sum_{i=1}^{n} \frac{9}{10^i} \) which is the sum of geometric series, so is equal to \( \frac{10^n - 1}{10^n} \), which in the limit is 1.

All this goes in one direction - given a decimal expansion we can define the corresponding real number as the limit of partial expansions. The converse (given a real number, find its expansion(s)) is more complicated. It is treated in Ross Section 16.

2 Limit infimum and limit supremum

Let \( s_n \) be a bounded sequence. The limit behaviour of \( s_n \) depends only on the “tail sets” \( \{s_n | n > N\} \). The sequence gets boxed in by \( v_N = \sup \{s_n | n > N\} \) and \( u_N = \inf \{s_n | n > N\} \). As \( N \) increases the box gets smaller. At “infinite \( N \)” the box becomes \( v = \lim v_N \) and \( u = \lim u_N \), called respecitvely limit superior written \( \lim \sup \) \( s_n \) and limit inferior written \( \lim \inf \) \( s_n \) (though I like to mentally call them “limit of supreums” and “limit of infimums” which is what they actually are). You may suspect that the box getting of “size 0” is the same as the sequence getting squeezed to a definite value, i.e. converging. We shall prove this shortly, just after discussing some examples.

Example 1: \( s_n = \frac{1}{n} \). Then \( v_N = \frac{1}{N} \), so \( \lim \sup \frac{1}{n} = \lim \frac{1}{N} = 0. \)
Example 2: \( s_n = 1 - \frac{1}{n} \). Then \( v_N = 1 \), so \( \lim \sup 1 - \frac{1}{n} = \lim 1 = 1 \)

Example 3: For a non-increasing sequence \( s_n \) with infimum \( i \) we have \( \lim \sup s_n = \lim s_N = i \).

Example 4: For a non-decreasing \( s_n \) with supremum \( s \) we have \( \lim \sup s_n = \lim s = s \).

Example 5: \( s_n = (-1)^n \). Then \( \lim \sup s_n = \lim 1 = 1 \)

Example 6: \( s_n = (-1)^n \frac{1}{n} \) we get \( v_N = \frac{1}{N+1} \text{ or } \frac{1}{N+2} \), depending on parity of \( N \), so \( 0 < v_N \leq \frac{1}{N+2} \) and by squeeze theorem \( \lim \sup s_n = \lim v_N = 0 \)

Example 7: \( s_n = (-1)^n (1 + \frac{1}{n}) \) so \( v_N = 1 + \frac{1}{N} \text{ or } 1 + \frac{1}{N+1} \), so \( 1 < v_N \leq 1 + \frac{1}{N+2} \) and \( \lim \sup s_n = \lim v_N = 1 \).

Next time we shall see how making the box of \( \lim \sup \) and \( \lim \inf \) size 0 is tantamount to getting \( s_n \) to converge, and use this to get a new criterion for convergence.