1 Limit theorems and examples III

Last time we saw limit of the reciprocal, and deduced limit of the ratios. We then defined divergence to plus and minus infinity and saw that for a positive sequence diverging to plus infinity is the same as having the sequence of inverses converge to 0. Then the fact that $\lim n^p = \infty$ implies $\lim \frac{1}{n^p} = 0$ (for $p > 0$).

We can also translate the limit theorems.

**Proposition 1.1.** Let $t_n$ and $s_n$ be sequences of positive numbers. If $\lim s_n = +\infty$ and $\lim t_n = t > 0$ then the sequence $t_n s_n$ diverges to plus infinity, $\lim t_n s_n = +\infty$.

We can deduce this to the theorems we have previously established by noting that if $x_n = \frac{1}{s_n}$ then $\lim x_n = 0$ and if $y_n = \frac{1}{t_n}$ then $\lim y_n = \frac{1}{t}$, so by product theorem (for normal, finite numbers) we get $\lim x_n y_n = 0 \cdot \frac{1}{t} = 0$. This is then equivalent to $\lim t_n s_n = \lim \frac{1}{x_n y_n} = +\infty$.

We note that this is an example of equivariance - we transformed $s_n$ and $t_n$ by taking the reciprocals, arguing about the resulting sequences and then transporting back the conclusions.

Observe that the same statement holds without requiring that $s_n$ and $t_n$ be positive. The point is that since their limits are positive the terms will be positive eventually, and the limit only cares about the tail part anyway.

**Proposition 1.2.** Let $t_n$ and $s_n$ be sequences of real numbers. If $\lim s_n = +\infty$ and $\lim t_n = t > 0$ then the sequence $t_n s_n$ diverges to plus infinity, $\lim t_n s_n = +\infty$.

**Proof.** There exists $N_1$ such that $s_n > 0$ for $n > N_1$. There exists $N_2$ such that $|t_n - t| < t/2$ for $n > N_2$, so that $-t/2 \leq t_n - t$ and $0 < t/2 < t_n$. Then for $N = \max\{N_1, N_2\}$ the sequences $s'_n = s_{n+N}$ and $t'_n = t_{n+N}$ satisfy the hypothesis of the previous version of the theorem. Hence $t'_n s'_n$ diverges to plus infinity; hence so does $t_n s_n$. \[\square\]
This is an example of the strategy of considering the infinite tail separately from the initial piece (with the bonus principle that "initial piece is usually of no importance").

See Ross for an alternative proof.

With this in mind, let’s get to some more examples.

Our examples are all of the form $a^b$ where either $a$ or $b$ (or both) depend on $n$.

**Example 1.3.** $\lim a^n = +\infty$ if $a > 1$.

The proof must rely on $a > 1$, for otherwise the conclusion would be false. So we write $a = 1 + b$ for $b > 0$. We want $(1 + b)^n > M$ for any $M$ for all $n$ large. It would help to have $(1 + b)^n$ bounded below by something that obviously grows to infinity...

**Proof.**

**Lemma 1.4** (Bernoulli inequality.). If $b > 0$ $(1 + b)^n \geq 1 + nb$

**Proof.** By induction. Base tautological. Step: $(1 + b)^n(1 + b) \geq (1 + nb)(1 + b) = 1 + (n + 1)b + nb^2 > 1 + (n + 1)b$, using the induction hypothesis in the first inequality. Alternatively: Use binomial formula (see Ross).

Now as $a > 1$ we have $a = 1 + b$ for some $b > 0$, so that $a^n = (1 + b)^n \geq 1 + bn > M$ for all $n > \frac{M}{b}$, so $a^n$ diverges to plus infinity.

**Example 1.5.** $\lim n^{\frac{1}{n}} = 1$.

We get the sequence $(1, \sqrt{2} \approx 1.414, \sqrt[3]{3} \approx 1.442, \sqrt[4]{4} \approx 1.414, \sqrt[5]{5} \approx 1.379, \ldots \frac{\sqrt[100]{100}}{100} \approx 1.047 \ldots 1]$ does seem to go to 1.

It seems to converge to 1 from above. That’s because $\frac{1}{n} \geq 1$, which is equivalent to $n \geq 1^n = 1$ which is true.

So what we need to show is that for any $\varepsilon > 0$ exists $N$ such that for $n > N$ we get $n^{\frac{1}{n}} - 1 < \varepsilon$. This is the same as $n^{\frac{1}{n}} < 1 + \varepsilon \text{ or } n < (1 + \varepsilon)^n$.

We can try using Bernoulli inequality again - $(1 + \varepsilon)^n > 1 + n\varepsilon$ but that is not above $n$ for small $\varepsilon$. But Bernoulli inequality is a “first degree in $\varepsilon$” expansion, with the Binomial theorem being the full expansion. We can use the full force, or we can go to degree 2.

**Lemma 1.6.** If $b > 0$ $(1 + b)^n \geq 1 + nb + \frac{n(n-1)b^2}{2}$ for all $n \geq 2$.

*Here is a fun puzzle: The sequence increases for the first two steps and seems to decrease afterwards. Can you see why? Here is another puzzle: Write 100 as sum of positive integers such that the product of those integers is maximized. Here’s a third puzzle: Why are the first two puzzles in the same footnote?*
Proof. Induction on \( n \geq 2 \) or the binomial formula (see Ross).

Now \((1 + \varepsilon)^n \geq 1 + \varepsilon n + \frac{n(n-1)}{2} \varepsilon^2 > \frac{n(n-1)}{2} \varepsilon^2\). To ensure that the last term \(\frac{n(n-1)}{2} \varepsilon^2\) to be bigger than \( n \) it suffices to make \( n > N = \frac{2}{\varepsilon^2} + 1 \), as little algebra will show.

Proof. Given \( \varepsilon > 0 \) let \( N = \frac{2}{\varepsilon^2} + 1 \). Then for \( n > N \) we have \((n - 1) > \frac{2}{\varepsilon^2}, \frac{n(n-1)}{2} \varepsilon^2 > n. \) Hence \((1 + \varepsilon)^n \geq \frac{n(n-1)}{2} \varepsilon^2 > n \) and \(1 + \varepsilon > n^{\frac{1}{n}}\) and \( n^{\frac{1}{n}} - 1 < \varepsilon \). Hence \( n^{\frac{1}{n}} \) converges to 1.

Here is our final “basic” example.

Example 1.7. \( \lim a^{\frac{1}{n}} = 1 \) for all \( a > 0 \).

Proof. We note that \( a^{\frac{1}{n}} \) behaves differently for \( a \geq 1 \) and \( a < 1 \), so we split into cases.

1) If \( a \geq 1 \) then \( a^{\frac{1}{n}} \geq 1^{\frac{1}{n}} = 1 \). We just need to ensure \( a^{\frac{1}{n}} \) gets all the way down to 1. But we know \( n^{\frac{1}{n}} \) does and for all \( a < n \) we have \( a^{\frac{1}{n}} < n^{\frac{1}{n}} \). Formally, if \( s_n = a^{\frac{1}{n}}, l_n = 1 \) and \( u_n = n^{\frac{1}{n}} \) then for all large enough \( n \) we have \( l_n \leq s_n < u_n \) and \( \lim l_n = \lim u_n = 1 \), so \( \lim s_n = 1 \) by the squeeze theorem, which is on the homework.

2) If \( a < 1 \) then \( \frac{1}{a} > 1 \) and \( \lim a^{\frac{1}{n}} = 1 \), so by limit of reciprocal theorem \( \lim a^{\frac{1}{n}} = 1 \) as well. This is equivariance again.

We can now combine this with limit theorems to compute things like \( \lim \frac{n(n^{\frac{1}{n}} + 5n)}{4n^2 + 2^{\frac{1}{n}}} \) as a routine matter. However their main importance will come later when we talk about series and radius of convergence for power series.