**QUESTION (i)**

\[ \limsup_{n \to \infty} s_n = \lim_{N \to \infty} \sup \{ s_n : n \geq N \} \]

**QUESTION (ii)**

For an ordered field, \( 0 < 1 \) (proved in HW1).

Thus, for every \( n \), \( n = n + 0 \leq n + 1 \), so we have
\[ 0 < 1 < 2 < 3 < 4 < \ldots \]

So if \( m \) and \( n \) are two elements in \( \mathbb{Z} \), \( 0, 1, 2, 3, \ldots \),
then one of them has to be "on the right" of the other.

Say \( m \) is on the right of \( n \), so
\[ m = n + 1 + \ldots + 1 \]
\[ m-n \text{ times.} \]

Then \( n < n + 1 < n + 1 + 1 < \ldots < n + 1 + 1 + 1 = m \]
\[ m-n \text{ times.} \]

Thus \( n \neq m \), meaning that \( n \neq m \).

Hence, all elements in the set \( \mathbb{Z} \) are distinct.

Since this set of elements is infinite, the ordered field has infinitely many elements.

**QUESTION (iii)**

If \( r = \frac{p}{q} \) is a rational solution, with \( p \) and \( q \) having no common factors, then \( p | 12 \) and \( q | 13 \) (from the Rational Zeros Theorem).

So \( p = 3, 1, 1 \), \( q = 3, 1, 1 \).

Thus \( \frac{p}{q} \in \{ \pm 1, \pm 3, \pm \frac{1}{3} \} \).
If \( r > 0 \), then \( 3r^3 + 2r^2 + 3r + 2 \geq 0 + 0 + 0 > 0 \), so the solution cannot be positive. Hence, \( r < 0 \).

\[
\begin{align*}
 r &= -1 \implies 3(-1)^3 + 2(-1)^2 + 3(-1) + 2 = -3 + 2 - 3 + 2 = -2 \neq 0 \\
 r &= -2 \implies 3(-2)^3 + 2(-2)^2 + 3(-2) + 2 = -24 + 8 - 6 + 2 = -20 \neq 0 \\
 r &= -1/3 \implies 3(-1/3)^3 + 2(-1/3)^2 + 3(-1/3) + 2 = -\frac{1}{9} + \frac{2}{3} - 1 + 2 = \frac{10}{3} \neq 0 \\
 r &= -2/3 \implies 3(-2/3)^3 + 2(-2/3)^2 + 3(-2/3) + 2 = -\frac{8}{9} + \frac{8}{3} - 2 + 2 = 0
\end{align*}
\]

Hence, \( r = -2/3 \) is the only rational solution of the equation.

**QUESTION (iv)**

Let's first prove that \( s_n > 1 \) for all \( n \).

We do this by induction:

For \( n = 1 \), \( s_1 = s > 1 \).

Assume it for \( n \), i.e. \( s_n > 1 \).

Then \( s_{n+1} = \sqrt{s_n} > \sqrt{1} = 1 \), so \( s_{n+1} > 1 \) too.

Hence, \( s_n > 1 \) for all \( n \).

Next, we show that \( s_n \leq 5 \) for all \( n \):

Do this again by induction:

For \( n = 1 \), \( s_1 = s \leq 5 \).

Assume it for \( n \), i.e. \( s_n \leq 5 \).

Then \( s_{n+1} = \sqrt{s_n} \leq \sqrt{5} \).

Now \( s > 1 \), so \( \sqrt{5} > 1 \) too \( \implies \sqrt{5}\sqrt{5} > \sqrt{5} \cdot 1 = 5 > 5 \).

Thus \( s_{n+1} \leq \sqrt{5} < 5 \).

Hence, \( s_n \leq 5 \) for all \( n \).

So \( s_n \) is bounded, \( 1 \leq s_n \leq 5 \) for all \( n \).
Now \( S_{n+1} = \sqrt{nS_n} < S_n \), since
\[
S_{n+1} > 1 \implies \sqrt{nS_n} > \sqrt{S_n} \cdot 1 \implies S_n < S_n^{1/2}
\]
Thus \( S_{n+1} < S_n \), i.e. the sequence is monotone decreasing.

We know that all monotone bounded sequences converge, so \( \lim S_n \) exists, say \( L = \lim S_n \).

Taking \( t_n = \sqrt{nS_n} \), then \( \lim t_n \) exists and equals \( L \).

Since \( S_n = t_n \), then \( \lim S_n = \lim t_n = L \) = \( \sqrt{L} \)

\[
L = \sqrt{L} \implies L = 0 \text{ or } L = 1.
\]

Now \( S_n \in [0, \infty) \) for all \( n \), so \( \lim S_n = [0, \infty) \) too.

Hence, \( L = 1 \), i.e. \( \lim S_n = 1 \).

**QUESTION (V) | SOLUTION 1.1**

We claim that \( \lim n^{1/n^2} = 1 \).

So let \( \varepsilon > 0 \). We want \( N \) such that for all \( n > N \):
\[
|n^{1/n^2} - 1| < \varepsilon
\]
\[
= |1 - \varepsilon| < n^{1/n^2} < 1 + \varepsilon
\]

Now \( 1 < n^{1/n^2} \) always, since \( 1 < n \), so \( 1 - \varepsilon < n^{1/n^2} \) is true despite \( n \).

We need \( n^{1/n^2} < 1 + \varepsilon \), i.e. \( n < (1 + \varepsilon)^{n^2} \) for \( n \) some \( N \).

Recall Bernoulli inequality:
For \( b > 0 \), \((1+b)^n \geq 1 + nb \).
Putting $\varepsilon$ instead of $h$ and $n^2$ instead of $n$, we get

$$(1 + \varepsilon)^n \geq 1 + n^2 \varepsilon$$

We want $(1 + \varepsilon)^n > n$, so we can require $1 + n^2 \varepsilon > n$ instead, or even $n^2 \varepsilon > 1$, since $1 + n^2 \varepsilon > n^2 > n$ then.

So this means that $\varepsilon > \frac{1}{n}$.

But from Archimedean property, we have $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, and for $n > N$ then, $\frac{1}{n} < \frac{1}{N} < \varepsilon$.

Passing to the formal proof, let $N$ be such that $\frac{1}{N} < \varepsilon$.

Then $n > N$ implies $\frac{1}{n} > \frac{1}{N} < \varepsilon$.

So $0 < n^2 \varepsilon < n^2 \varepsilon + 1 \leq (1 + \varepsilon)^n$.

Bernoulli inequality.

So for $n > N$, $n < (1 + \varepsilon)^n = e^{n \ln(1 + \varepsilon)}$.

So for all $n > N$,

$$1 \leq e^{n \ln(1 + \varepsilon)} = 1 + n \ln(1 + \varepsilon)$$

$$\Rightarrow -\varepsilon < \frac{n \ln(1 + \varepsilon)}{n} < \varepsilon$$

$$\Rightarrow \frac{\ln(1 + \varepsilon)}{n} < \varepsilon$$

Hence, $\lim_{n \to \infty} \frac{\ln(1 + \varepsilon)}{n} = 0$.

[Solution 2]

We can make use of the fact that $\lim_{n \to \infty} a_n = 1$ (proved in class).

Let $a_n = 1$ (the constant sequence).

$b_n = \frac{n}{n^2}$ and $c_n = \frac{n}{n^2}$.

Clearly, $1 \leq n$ so $\frac{1}{n^2} \leq \frac{n}{n^2}$, i.e. $1 \leq n^2 = 1 + b_n, \forall n$.

Also $\frac{1}{n^2} \leq \frac{1}{n}$, so $\frac{n}{n^2} \leq \frac{1}{n}$ too, i.e. $b_n \leq c_n, \forall n$.

Now $\lim_{n \to \infty} a_n = 1$, $\lim_{n \to \infty} c_n = 1$, and $a_n \leq b_n \leq c_n$ for all $n$. 

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By Squeeze Theorem, \( \lim b_n \) exists and it equals 1. Thus \( \lim a_n^{1/n} = 1 \).

**QUESTION (vi)**

Define \( V_n = \sup \{ s_n : n > N \} \) and \( V' = \sup \{ |s_n| : n > N \} \).

For a fixed \( n > N \) we have \( s_n \leq |s_n| \leq \sup \{ |s_n| : n > N \} = V' \) \( \Rightarrow s_n \leq V' \).

Now \( n > N \) was arbitrary, so \( s_n \leq V' \) for all \( n > N \). Then \( s_n \leq V' \) for all \( n \).

Hence, \( \sup \{ s_n : n > N \} \leq V' \) too, i.e. \( V_n \leq V' \).

Now consider \( u_n = V' - V_n \). Then \( 0 \leq u_n \) for all \( n \).

We have that \( s_n \) is a bounded sequence, hence so is \( |s_n| \), and therefore \( u_n = \sup \{ s_n : n > N \} \) and \( V_n = \sup \{ |s_n| : n > N \} \) are both real numbers for every \( N \).

\[ \limsup \{ s_n \} \text{ and } \limsup \{ |s_n| \} \text{ both exist and are real numbers} \]

(From the boundedness of \( s_n \) and \( |s_n| \)), so \( \lim V_n \) and \( \lim V' \) exist too.

Hence, \( \lim u_n \) exists and equals \( \lim V' - \lim V_n \) (remember, \( u_n = V' - V_n \)). On the other hand, \( 0 \leq u_n \) for all \( n \), so \( \lim u_n \neq 0 \) too (this question).

\[ 0 \leq \lim u_n = \lim V' - \lim V_n \]
\[ = \limsup |s_n| - \limsup s_n \]
\[ = \limsup s_n \leq \limsup |s_n| \]
(a) Let \( S_n = \frac{(-1)^n}{n} \).

Then

- \((-1)^n \leq \frac{1}{n} \leq S_n \) for all \( n \), so \( S_1 \leq S_n \). \( \forall n \Rightarrow \inf S_n = -1 \)
- For \( n > 2 \), \( \frac{1}{n} > \frac{-1}{n} \), so \( S_2 \geq S_n \).
  
  also \( S_1 = -1 < S_2 \), so for all \( n \), \( S_2 \geq S_n \) \( \Rightarrow \sup S_n = S_2 = \frac{1}{2} \)
- \( \lim S_n = 0 \). Let \( \varepsilon > 0 \). Then \( \exists N \) such that \( \frac{1}{N} < \varepsilon \)
  
  for this \( N \), for all \( n > N \), we have
  
  \[ |S_n - 0| = \left| \frac{(-1)^n}{n} \right| = \frac{1}{n} < \frac{1}{N} < \varepsilon \]
  
  i.e. \( |S_n - 0| < \varepsilon \)
  
  Hence, \( \lim S_n = 0 \).

So \( -\infty < -1 < 0 < \frac{1}{2} < +\infty \)

\( \inf \) \( \lim \) \( \sup \).

(b) Let

\[ S_n = \begin{cases} \sqrt{n} & \text{if } n \text{ is even} \\ -n & \text{if } n \text{ is odd} \end{cases} \]

Then

- \( \inf \{ S_n : n > N \} = -\infty \) since \( S_n \) is not bounded below.
  
  (In fact, for every \( M \), there exists an odd integer \( N > M \) such that \( S_N < M \). Hence, for all \( M \), \( \exists N' \) such that \( S_{N'} < M \).)

Hence,

\[ \liminf S_n = \lim \inf \{ S_n : n > N \} = +\infty \]

- \( \sup \{ S_n : n > N \} = \sqrt{2} \) because there is some even integer \( N' > M \) and \( S_{N'} = \sqrt{2} \).
  
  Also for all \( n > N \), \( S_n \leq \sqrt{2} \).

Thus, \( \sup \{ S_n : n > N \} = \sqrt{2} \).

Hence,

\[ \limsup S_n = \lim \sup \{ S_n : n > N \} = \frac{\sqrt{2}}{2} \]