Total possible points: 220

1. (10 point) Prove $(2 + \sqrt{2})^{1/2}$ is irrational.

2. (30 points) Decide (with proof) whether the following series converge.
   1. $\sum_{n=2}^{\infty} \frac{1}{n^{2-n}}$
   2. $\sum \frac{n}{2^n}$
   3. $\sum (-1)^n \frac{1}{\cos n}$

3. (30 points) Let $f(x)$ be equal to $x^{17}$ for all $x$ that are of the form $x = \sqrt{2^p}q$ for $q \in \mathbb{Z}$, $q \in \mathbb{N}$, that is, for $x$ that are multiples of rational. Further assume $f(x)$ is continuous. Prove $f(x) = x^{17}$ for all $x$

4. (20 points) Suppose $\lim_{n \to \infty} (a_{n+1} - a_n) = 0$. Is the sequence $a_n$ convergent?

5. Suppose $a_n = \frac{1}{n^2}$ for even $n$, and $a_n = \frac{1}{3^n}$ for odd $n$.
   1. (8 pts) Show that the ratio test fails to determine if the series converges.
   2. (7 pts) Show that the series converges.

6. (20 pts) Is $\sin(\frac{1}{x})$ uniformly continuous on $(0,1)$? Give full reasoning.

7. (20 pts) Prove $\sum \frac{1}{m^2} 2^{nx}$ converges uniformly on $[0,1]$. Find the sum.

8. (20 pts) Suppose $a_n \geq 0$ and $\sum a_n$ diverges. Show $\sum \frac{a_n}{1+a_n}$ diverges. Hint: Consider two cases, ($\lim a_n = 0$) and (NOT $\lim a_n = 0$).

9. (20 pts) Suppose $f_n(x)$ converges to $f(x)$ uniformly and suppose for some fixed $a$ we have $\lim_{x \to a} f_n(x) = a_n$ (that is, the limit of $f_n$ at $a$ exists for each $n$ and is equal to $a_n$). Show $\lim_{n \to \infty} a_n$ exists.
   Note: Do NOT assume $f_n$ or $f$ continuous.

10. (15 pts) Find radius of convergence $R$ for the following power series.
    1. $\sum \frac{x^{2^n}}{3^n}$
    2. $\sum \frac{x^n}{n^n}$

11. (20 pts) Suppose a sequence of continuous functions $f_n(x) : [0,1] \to \mathbb{R}$ converges uniformly to $f(x)$. Show $\sup\{f_n(x)|n \in \mathbb{N}, x \in [0,1]\}$ is finite.

Extra practice problem:

Note: This is probably too long for an actual final.

12. Let $s_n$ be a sequence of real numbers. Let $t_n = s_n^2$.
   1. (15 pts) Prove $\sup t_n = \max\{\sup s_n, \inf s_n\}$.
   2. (15 pts) Prove $\lim \sup t_n = \max\{\lim \sup s_n, \lim \inf s_n\}$. 