PROBLEM 1.

1. \( \sum_{n=1}^{\infty} \frac{1}{n+1} \) diverges because \((-1)^n \frac{1}{n} x \to 0.

\[
\text{(In fact, } |(-1)^n \frac{1}{n} | \to 1)\]

2. \( \sum_{n=1}^{\infty} \frac{2n+1}{3n+1} \) converges by comparison test,

\[
2^{n+1} < 2^n + 2^n \quad \text{and} \quad 3n+1 > 3^n, \quad \text{so}
\]

\[
\frac{2^{n+1}}{3n+1} < \frac{2 \cdot 2^n}{3^n} = 2 \cdot \left(\frac{2}{3}\right)^n \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n
\]
forms a convergent geometric series.

3. \( \sum_{n=1}^{\infty} n \log n \) converges by integral test.

Let \( f(x) = x \log x \). Both \( x \) and \( \log x \) are increasing on \((1, \infty)\).

\[
\text{Thus, } f(x) \text{ is decreasing on } [n, n+1]. \quad \text{Also } f(n) = an
\]

\[
\int_{1}^{\infty} f(x) \, dx = \int_{1}^{\infty} x \log x \, dx = \int_{1}^{\infty} e^{u} \, du
\]

\[
\left. \frac{e^{u}}{u} \right|_{1}^{\infty} = \lim_{u \to \infty} \left( \frac{e^{u}}{u} - \frac{1}{e} \right) = \infty
\]

\[
\Rightarrow \sum_{n=1}^{\infty} n \log n \text{ converges}
\]

PROBLEM 2.

For all \( s, t \in SUT \), so \( s \leq \sup SUT \).

Since this holds for all \( s, t \), then \( \sup S \leq \sup SUT \).

Likewise, \( \sup T \leq \sup SUT \).

\[
\Rightarrow \max \{ \sup S, \sup T \} \leq \sup SUT
\]

On the other hand, let \( x \in SUT \).

If \( x \in S \), then \( x \leq \sup S \leq \max \{ \sup S, \sup T \} \).

If \( x \in T \), then \( x \leq \sup T \leq \max \{ \sup S, \sup T \} \).

Hence, \( x \leq \max \{ \sup S, \sup T \} \).

Since this holds for all \( x \in SUT \), then

\[
\sup SUT \leq \max \{ \sup S, \sup T \}
\]
(2) Let \( x \in S \cap T \). Then \( x \in S \) and \( x \in T \).

\[
\begin{align*}
\sup S, \sup T & \leq x \leq \min \{\sup S, \sup T\} \\
\text{This is true for all } x \in S \cap T, \quad \Rightarrow \\
\sup S \cap T & \leq \min \{\sup S, \sup T\}
\end{align*}
\]

(3) Let \( S = \{1, 2, 3\} \), \( T = \{1, 3\} \).

Then \( \sup \{S \cap T\} = \sup \{1, 3\} = 3 \)

but \( \min \{S \cap T\} = \min \{1, 3\} = 1 \)

PROBLEM 3

Let \( P(x) = x^3 \). Polynomials are continuous, so \( P \) is continuous.

\( P \) is strictly increasing on \( \mathbb{R} \), and \( \mathbb{R} = (-\infty, \infty) \) is an interval.

Then \( f^{-1}(y) = y^{1/3} \) represents a continuous function with domain \( f(\mathbb{R}) = \mathbb{R} \).

Hence, \( f^{-1}(y) = y^{1/3} \) is a continuous function on \( \mathbb{R} \).

Then \( S_n \to S \Rightarrow \sqrt[3]{S_n} \to \sqrt[3]{S} \), i.e. \( S_n \to S \).

and likewise \( \sqrt[3]{T_n} \to \sqrt[3]{T} \).

Finally, \( \sqrt[3]{S_n} + \sqrt[3]{T_n} \to \sqrt[3]{S} + \sqrt[3]{T} \).

PROBLEM 4

(1) Let \( x_0 \) be such that \( g(x_0) = 0 \).

Let \( \varepsilon > 0 \). Since \( g \) is continuous, \( \exists \delta > 0 \) s.t. \( |x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \varepsilon \).

\[
|g(x) - g(x_0)| = |g(x) - 0| = |g(x)| < \varepsilon
\]

With \( \delta \) as above, let \( |x - x_0| < \delta \).

Clearly \( f(x_0) = 0 \). So:

- If \( x \) is rational, \( f(x) = g(x) \), \( \Rightarrow \)
  \[
  |f(x) - f(x_0)| = |g(x) - 0| = |g(x)| < \varepsilon
  \]

- If \( x \) is irrational, \( f(x) = 0 \), \( \Rightarrow \)
  \[
  |f(x) - f(x_0)| = |0 - 0| = 0 < \varepsilon
  \]

Hence, \( |x - x_0| < \varepsilon \) \( \Rightarrow |f(x) - f(x_0)| < \varepsilon \).

\( \therefore f \) is continuous at \( x \).
(2) Let \( x_0 \) be such that \( g(x_0) \neq 0 \), and assume \( f \) continuous at \( x_0 

Both rationals and irrationals are dense in \( \mathbb{R} \), so

- \( \exists \) a sequence of rationals \( r_n \) s.t. \( r_n \to x_0 \)

From continuity of \( f \),

\[
f(r_n) \to f(x_0)
\]

But \( f(r_n) = g(r_n) \), and \( r_n \to x_0 \), \( g \) continuous, so

\[
f(r_n) = g(r_n) \to g(x_0)
\]

\( \Rightarrow \) By uniqueness of limit, \( f(x_0) = g(x_0) \) --- (1)

- \( \exists \) a sequence of irrationals \( s_n \) s.t. \( s_n \to x_0 \)

Then \( f(s_n) \to f(x_0) \).

But \( f(s_n) = 0 \), \( \forall n \) \( \Rightarrow f(x_0) = 0 \) --- (2)

Hence, \( f(x_0) = 0 \) & \( f(x_0) = g(x_0) \) \( \Rightarrow g(x) = 0 \), contradicting

the fact that \( g(x_0) \neq 0 \)

\( \Rightarrow f \) is discontinuous at \( x_0 \)

**Problems**

\( f \) is uniformly continuous on \( (0,1) \) iff \( f \) can be extended
to a continuous function \( \tilde{f} \) on \( [0,1] \).

So consider

\[
f(x) = \left\{ \begin{array}{ll}
x \sin \frac{1}{x} & \text{if } x \in (0,1) \\
0 & \text{if } x = 0
\end{array} \right.
\]

Clearly \( \tilde{f} \) extends \( f \) and \( \tilde{f} \) is continuous in \( (0,1) \),
because both \( x \) and \( \sin \frac{1}{x} \) are such that

\( g(x) = \sin x \) is continuous everywhere, \( h(x) \) continuous on \( (0,1) \)

\( \Rightarrow (g \circ h)(x) = \sin \frac{1}{x} \) is continuous on \( (0,1) \)
At \( x = 0 \), let \( \varepsilon > 0 \). Then, for \( \delta = \varepsilon / 2 \), if \( |x - 0| = |x| < \delta \),

\[
0 < \delta = \varepsilon / 2 \leq |x| < \delta \Rightarrow |x - 0| = |x| < \delta
\]

\[
|\tilde{f}(x) - \tilde{f}(0)| = |x \sin \frac{1}{x} - 0| = |x| |\sin \frac{1}{x} - 0| = |x| \leq |x + 1| \leq 1 \leq \varepsilon
\]

So \( |x - 0| < \delta \Rightarrow |\tilde{f}(x) - \tilde{f}(0)| < \varepsilon \)

\[\Rightarrow \tilde{f} \text{ is continuous at } 0.\]

So \( \tilde{f} \) is a continuous extension of \( f \) to \([0, 1]\)

i. \( f \) is uniformly continuous on \((0, 1)\)

**Problem 6**

1. \[
\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1-x} \rightarrow \frac{1-0}{1-x} = \frac{1}{1-x} \text{ as } n \to \infty \text{ for } |x| < 1
\]

So \( |x| < 1 \Rightarrow \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \)

Then, for \( x = \frac{1}{2} \), \( \sum_{k=0}^{\infty} \frac{1}{2^k} = 1 - \frac{1}{2} = 2 \)

2. \[
\sum_{k=1}^{n} kx^{k-1} = \left( \sum_{k=0}^{n} x^k \right)' = \left( \frac{1-x^{n+1}}{1-x} \right)' = \frac{-n x^n (1-x) - (1)(1-x')}{(1-x)^2} = \frac{-nx^n + nx^n + 1 - x^n}{(1-x)^2} = \frac{0+0+1-0}{(1-x)^2} = \frac{1}{(1-x)^2} \text{ as } n \to \infty \text{ for } |x| < 1
\]

Thus \( \sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2} \) for \( |x| < 1 \)

For \( x = \frac{1}{2} \), \( \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{2^0} = 4 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} \cdot 4 = 2 \)

\[
= \sum_{n=0}^{\infty} \frac{n+1}{2^n} = \frac{0}{2^0} + \sum_{n=1}^{\infty} \frac{n}{2^n} + \sum_{n=0}^{\infty} \frac{1}{2^n} = 0 + 2 + 2 = 4
\]

= 2 from (1)
PROBLEM 7

For all $x \in \mathbb{R}$,
\[
\begin{align*}
\frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} 
&= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \\
&\text{so by Weierstrass M-test,} \\
\frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \sin((2k+1)\pi x) 
&\text{converges uniformly on } \mathbb{R}.
\end{align*}
\]

PROBLEM 8

Let $f'(a)$ exist and $\epsilon > 0$. Since
\[
\frac{f(x) - f(a)}{x - a} \to f'(a),
\]
there exists $\delta$ such that $0 < |x - a| < \delta \Rightarrow \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \frac{\epsilon}{2}.
\]

Let $\delta = \min \{ \delta, \frac{\delta}{2|f'(a)| + 1} \} > 0$.

Then, for all $x$ such that $|x - a| < \delta$,

- if $x = a$, $|f(x) - f(a)| = |f(a) - f(a)| = 0 < \delta$.

- if $x \neq a$, $|x - a| < \delta \Rightarrow \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \frac{\epsilon}{2}.
\]

\[
\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \frac{\epsilon}{2}.
\]

\[
\Rightarrow |f(x) - f(a)| < \frac{\epsilon}{|f'(a)| + 1}.
\]

\[
|f(x) - f(a)| < (|f'(a)| + 1) \cdot \frac{\epsilon}{|f'(a)| + 1}.
\]

\[
< (|f'(a)| + 1) \cdot \frac{\epsilon}{|f'(a)| + 1}.
\]

\[
= \frac{\epsilon}{|f'(a)| + 1}.
\]

\[
\Rightarrow |x - a| < \delta \Rightarrow |f(x) - f(a)| < \delta.
\]

So $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \delta = \epsilon$.

$f$ is continuous at $a$. 

\[\text{page 5}\]
PROBLEM 9.

(1) \( \sum \frac{x^n}{n^2 + 2n} = \sum \frac{1}{n^2 + 2n} x^n \)

\[
\limsup \left( \frac{1}{n^2 + 2n} \right)^{1/n} = 1, \text{ so the radius of convergence is } R = \frac{1}{2} = 1.
\]

At \( R = 1 \),

\[
\sum \left( \frac{1}{n^2 + 2n} \right)^n < \sum \frac{1}{n^2} \quad \text{and} \quad \sum \frac{1}{n^2} \quad \text{converges} \Rightarrow \sum \frac{1}{n^2} \quad \text{converges}
\]

At \( R = -1 \),

\[
\sum \frac{(-1)^n}{n^2 + 2n} = \sum (-1)^n \frac{1}{n^2 + 2n}, \quad \text{and} \quad \frac{1}{n^2 + 2n} \quad \text{is a decreasing sequence with limit 0, so by alternating series test,}
\]

\[
\sum \frac{(-1)^n}{n^2 + 2n} \quad \text{converges}
\]

(2) \( \sum x^n = \sum a_n x^n \), where \( a_n = 1 \) if \( x = n^2 \) and 0 otherwise.

\[
= 1 \limsup a_n^{1/n} = 1. \quad \text{Hence, } R = 1
\]

At \( R = 1 \), \( \sum x^n = \sum 1 \), \( \sum x^n \) diverges.

At \( R = -1 \), the partial sums of \( \sum (-1)^n \) are \( S_n = \sum_{k=0}^{n} (-1)^k = 1 - (-1) - (-1)^2 \), so \( S_n = 0 \) if \( n \) odd and \( S_n = 1 \) if \( n \) even. Hence, \( S_n \) does not converge \( \Rightarrow \) \( \sum x^n \) does not converge for \( x = -1 \).

PROBLEM 10

Let \( \varepsilon > 0 \). \( P_n \to P \) uniformly, so \( \exists N \) such that

\[
|P_n(x) - P(x)| < \varepsilon/3 \quad \text{for all } x \in (0, 1)
\]
So let \( n > N \) be any integer.

\( f_n \) is uniformly continuous, so \( \exists \delta > 0 \) s.t.
\[
|x - y| < \delta \implies |f_n(x) - f_n(y)| < \varepsilon/3 \quad \text{(8)}
\]

Then, for this \( \delta \), \( |x - y| < \delta \)

\[
|f(x) - f(y)| = |(f(x) - f_n(x)) + (f_n(x) - f_n(y)) + (f_n(y) - f(y))| \\
\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\]

because
\[
n > N \implies |f(x) - f_n(x)| < \varepsilon/3 \quad \text{and} \quad |f_n(y) - f(y)| < \varepsilon/3 \quad \text{for} \quad x, y \in (a, 1) \quad \text{and}
\quad |f_n(x) - f_n(y)| < \varepsilon/3 \quad \text{from} \quad (8)
\]

Hence, \( |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \)

\( \Rightarrow \) \( f \) is uniformly continuous on \( (a, 1) \)

**Problem 11.**

Let \( f(x) = 0 \). Then \( f_n \to f \) pointwise.

Let \( x \in \mathbb{R} \). For all \( n > 1 + x \), \( x - n < x - 1 + x - 1 < 0 \), so
\[
x - n \not\in [0, 1] \implies f_n(x) = f(x - n) = 0 \quad \text{for all} \quad n > 1 + x
\]

Hence, \( f_n(x) \to 0 = f(1) \) as \( n \to \infty \).

10. \( f_n \to f \) pointwise.

The convergence is not uniform: for all \( n > 1 \),
\[
|f_n(n) - f(n)| = |f(n - n) - f(n)| = |f(1) - f(0)| = 1 - 0 = 1
\]

So there is no \( N \) such that for all \( n > N \) and all \( x \in \mathbb{R} \),
\[
|f_n(x) - f(x)| < \frac{1}{n}
\]