• Complete the problems on the back of this cover sheet. In order to receive full credit, please show all the steps. You may use any result proved in class or the text, unless the problem asks you to reprove such a result, in which case you may use all the results logically preceding it. Be sure to clearly state the results you are using before using them, and to verify that all hypotheses are satisfied.

• This is a closed-book, closed-notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.

• You have 3 hours. I will signal the times between which you are permitted to be writing, including anything on this cover sheet, and to have the exam booklet open. During these times, the exam and all papers must remain in the testing room. When you are finished, you must hand your exam paper to me.

• Use the blue book to write your solutions; make sure to write your name on it. Please indicate which problem you are solving. If you need extra paper for scratch work, use other pages of the blue book, or ask for another blue book.

• Don’t panic. I will translate points to letter grades based on what I think you understand. Show me you understand things. Partial credit will be given generously, but good writing will make it easier to implement this generosity.

• Please sign the following:

   “On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

   Signature: __________________________

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![Comic Strip](image)
1. (25 pts) Decide (with proof) whether the following series converge.

1. \( \sum (-1)^n \frac{1}{n^{1/3}} \)
2. \( \sum \frac{2^n + 1}{3^n} \)
3. \( \sum \frac{1}{n(\log n)^2} \)

2. Let \( S \) and \( T \) be bounded subsets or \( \mathbb{R} \)

1. (10 pts) Prove \( \sup S \cup T = \max \{ \sup T, \sup S \} \)
2. (10 pts) Prove \( \sup \{ S \cap T \} \leq \min \{ \sup T, \sup S \} \)
3. (5 pts) Find example where \( \sup \{ S \cap T \} \neq \min \{ \sup T, \sup S \} \)

3. (15 pts) Let \( s_n \) and \( t_n \) be sequences of numbers converging to \( s \) and \( t \) correspondingly. Prove

\[ \sqrt{s_n} + \sqrt{t_n} \]

4. Let \( g(x) \) be a continuous function. Further, let \( f(x) \) be defined by \( f(x) = 0 \) for all irrational \( x \), \( f(x) = g(x) \) for all rational \( x \).

1. (15 pts) Prove \( f(x) \) is continuous at all points where \( g(x) = 0 \)
2. (15 pts) Prove \( f(x) \) is discontinuous at all points where \( g(x) \neq 0 \).

5. (15 pts) Is \( x \sin \left( \frac{1}{x} \right) \) uniformly continuous on \( (0, 1) \)? Give full reasoning.

6. 1. (5 pts) Compute \( 1 + \frac{1}{2} + \frac{1}{4} + \ldots = \sum \frac{1}{2^n} \)
2. (10 pts) Compute \( 1 + 2/2 + 3/4 + 4/8 + 5/16 + \ldots = \sum \frac{n+1}{2^n} \)

7. (15 pts) Fourier series of the triangle wave is given by

\[
\frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin((2k+1)\pi x) = \frac{8}{\pi^2} \left( \sin(\pi x) - \frac{1}{9} \sin(3\pi x) + \frac{1}{25} \sin(5\pi x) + \ldots \right). 
\]

Prove that it converges uniformly on \( \mathbb{R} \).

8. (15 pts) Recall that when it exists, the derivative of a function \( f \) at \( a \) is \( f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \). Show that if \( f'(a) \) exists, then \( f \) is continuous at \( a \).

9. (20 pts) Find radius of convergence \( R \) for the following power series. Does the power series converge at \( R \)? At \(-R\)?

\[
\sum \frac{x^n}{n^2 + 2n} \\
\sum x^{n^2} 
\]

10. (10 pts) Suppose \( f_n \) is a sequence of functions, with each \( f_n \) uniformly continuous on \((0, 1)\). Suppose further that \( f_n \) converges uniformly to \( f \). Show that \( f \) is uniformly continuous on \((0, 1)\).

11. (15 pts) Let \( f_0(x) = 1 \) if \( x \in [0, 1] \) and \( f_0(x) = 0 \) if \( x \notin [0, 1] \).

Let \( f_n(x) = f_0(x - n) \). Does the sequence of functions \( f_n \) converge (pointwise)? Does it converge uniformly?