1. In class we defined multiplicative or Dirichlet characters \((\mod q)\). In this problem you will find the analogous additive characters \((\mod q)\). These are functions \(\psi : \mathbb{Z} \to \mathbb{C}\), not identically zero, such that \(\psi\) is periodic with period \(q\) (i.e. \(\psi(n+q) = \psi(n)\)) and also \(\psi(m+n) = \psi(m)\psi(n)\). Describe all the additive characters \((\mod q)\). Formulate and prove the orthogonality relations for these additive characters.

2. Let \(a(1), a(2), \ldots\) be a sequence of non-negative real numbers such that \(\sum_{n=1}^{\infty} a(n)\) diverges. Suppose that the power series

\[
f(z) = \sum_{n=1}^{\infty} a(n)z^n
\]

converges for all complex numbers \(|z| < 1\), and that

\[
\lim_{r \to 1^-} f(re^{i\theta})
\]

exists, and is finite, for every \(0 < \theta < 2\pi\). Prove that for any progression \(a \pmod q\),

\[
\sum_{n=1}^{\infty} a(n)
\]

diverges. Hint: Problem 1.

3. Let \(p\) be a prime and let \(\chi\) be a non-principal Dirichlet character \((\mod p)\). Define the order of \(\chi\) to be the least exponent \(\ell\) such that \(\chi^\ell = \chi_0\) where \(\chi_0\) is the principal character. If \(\chi\) has order \(\ell\) show that the values \(\chi(n)\) for \((n, q) = 1\) are \(\ell\)-th roots of unity. If \(g\) is a primitive root \((\mod p)\) then show that \(\chi(g)\) is a primitive \(\ell\)-th root of unity. If \(\chi(n)\) is a primitive \(\ell\)-th root of unity does it necessarily follow that \(n\) is a primitive root \((\mod p)\)? (Recall that a primitive \(n\)-th root of unity is a number of the form \(e^{2\pi i a/n}\) where \((a, n) = 1\).)

4. Let \(p\) be an odd prime. How many real (as opposed to complex) characters are there \((\mod p)\)? How many real characters are there \((\mod p^\alpha)\)? How many real characters are there \((\mod 2^\alpha)\)? How many real characters are there \((\mod q)\) for a general composite number \(q\)?