1. Let $|t| \geq 1$. Prove that
\[
\zeta(1 + it) = \sum_{n \leq |t|} \frac{1}{n^{1+it}} + O(1).
\]
Conclude that
\[
|\zeta(1 + it)| \leq \log |t| + O(1).
\]

2. For $|t| \geq 1$ obtain, as in problem 1, an approximation for $\zeta'(1 + it)$, and deduce an estimate for $|\zeta'(1 + it)|$.

3. Prove, using the Euler product or otherwise, that for $\sigma > 1$
\[
\zeta(\sigma) \geq |\zeta(\sigma + it)| \geq \frac{\zeta(2\sigma)}{\zeta(\sigma)}.
\]

4. Let $\|x\| := \min_{n \in \mathbb{Z}} |x - n|$ denote the distance between $x$ and the nearest integer. Suppose you are given real numbers $\alpha_1, \ldots, \alpha_K$. For any integer $N \geq 1$ prove that there exists $n$ with $1 \leq n \leq N^K$ such that $\|n\alpha_j\| \leq 1/N$ for each $j = 1, \ldots, K$. Hint: Divide the $K$-dimensional hypercube $[0,1)^K$ into cuboids with side-length $1/N$. For each $0 \leq n \leq N^K$ associate a point in this hypercube; use the pigeonhole principle.

5. Let $\sigma > 1$ be fixed, and suppose $\epsilon > 0$ is given. Show that there exists a non-zero real number $T = T(\sigma, \epsilon)$ such that
\[
|\zeta(\sigma + it) - \zeta(\sigma + it + iT)| \leq \epsilon
\]
for all real numbers $t$. In other words, the zeta-function on the line Re$(s) = \sigma$ is almost periodic, and $T$ is called an $\epsilon$-almost period. Hint: First show that $\zeta(\sigma + it)$ is well approximated by a suitable truncation of the Euler product, and then use the result of problem 4.