Math 115,  fall 2009,  Review problems,

1. For \( 0 \leq k \leq n \), define \( \binom{n}{k} \) as the number of distinct \( k \)-element subsets of the set \( \{1, 2, \ldots, n\} \).
   a. Prove that \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \).
   b. Prove by induction that \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).

2. a. Knowing that \( \mathbb{R} \) is complete, prove that \( \mathbb{R}^3 \) endowed with the metric \( \|x-y\| = \sqrt{\sum_{i=1}^{3} (x_i - y_i)^2} \) (where \( x = (x_1, x_2, x_3) \) and \( y = (y_1, y_2, y_3) \)) is complete: every Cauchy sequence in \( \mathbb{R}^3 \) has a limit.
   b. Prove: Every closed and bounded set \( E \subset \mathbb{R}^3 \) is compact (i.e., every open cover of \( E \) has a finite subcover).

3. Assume \( a > 0 \), \( p \) an arbitrary real number. Prove: \( \lim_{n \to \infty} n^p (1 + a)^{-n} = 0 \).
   Hint: Use the binomial expansion of \( (1 + a)^n \)
   Answer: By the binomial expansion, \( (1 + a)^n > \binom{n}{k} a^k \) when \( n > k \).
   Take \( k > p + 1 \) and remember \( \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \) so that for the constant \( C = a^{-k}2^k \) and \( n > 2k \) we have \( (1 + a)^n > C^{-1}n^k \) and \( n^p (1 + a)^{-n} < Cn^{-p-k} < \frac{C}{n} \) which converges to zero.

4. Prove or disprove (by giving a counter-example) each of the following statements: (the sequences \( \{a_n\} \) and \( \{b_n\} \) are assumed to be bounded sequences of real numbers)
   a. \( \limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n \).
   b. \( \limsup a_n b_n \leq \limsup a_n \limsup b_n \).
   c. \( \liminf(a_n + b_n) \leq \liminf a_n + \liminf b_n \).
   d. \( \liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n \).
   Answer:
   a. True: For every \( \epsilon > 0 \), appropriate \( N \) and \( j > N \), \( a_j < \limsup a_n + \epsilon \), \( b_j < \limsup b_n + \epsilon \) so that \( a_j + b_j < \limsup a_n + \limsup b_n + 2\epsilon \).
   b. False (unless, e.g., both sequences are positive): \( a_n = 1 + (-1)^n \) and \( b_n = -1 \). Then \( \limsup a_n = 2 \), \( \limsup b_n = -1 \) so their product is \(-2\) while \( \limsup a_n b_n = 0 \). of the limits
   c. False: \( a_n = 1 + (-1)^n \), \( b_n = 1 - (-1)^n \). We have \( \liminf a_n = \liminf b_n = 0 \), while \( \liminf(a_n + b_n) = 1 \).
   d. True. This is a. in which everything is multiplied by \(-1\).

5. Let \( f \) be defined and differentiable in \([a,b]\), and assume that \( f'(x) > 0 \) for all \( x \in [a,b] \).
   a. Prove that \( f \) is strictly monotone on \([a,b]\).
   b. Prove that the inverse function \( f^{-1} \) is differentiable on \([f(a), f(b)]\);
   c. what is the derivative of \( f^{-1} \)?
d. Is the derivative of $f^{-1}$ necessarily bounded?

**Answer:** No. $f'$ can be positive everywhere but not bounded away from 0.

Example: $f(x) = x + \sum_{n=5}^{\infty} \varphi_n(x)$ where $\varphi_n$ is differentiable, $\varphi_n(x) = 0$ outside the interval $I_n = \{x: |x - n^{-1}| < n^{-5}\}$ and range of $\varphi_n'$ is $[n^{-2} - 1, 2]$. $\varphi_n$ has no effect on $f''(0)$, but $\liminf_{x \to 0} f'(x) = 0$

6. Let $f$ be differentiable in $[0, 2]$. Assume $f(0) = 0$, $f(1) = -1$ and $f(2) = 3$. Prove that the equation $f'(x) = 0$ has at least one solution in $(0, 2)$.

**Answer:** $\min_{0 < x < 2} f(x)$ is attained at a point in $(0, 2)$.

7. Assume that $f$ is continuous on $[0, 1]$, $f(0) = 2$, $f(1) = 1/3$. Prove that for some $x \in [0, 1]$ we have $f(x) = 2x$.

**Answer:** $g(x) = f(x) - 2x$ is positive for $x = 0$ and negative for $x = 1$.

8. Let $p > 0$, $m \in \mathbb{N}$ (a positive integer). Prove the following upper and lower bounds for $\sum_{m+1}^{2m} n^{-p}$:

$$2^{-p} m^{1-p} \leq \sum_{m+1}^{2m} n^{-p} \leq m^{1-p}.$$  

**(a)** Use these to prove that $\sum_{1}^{\infty} n^{-p}$ is convergent if $p > 1$, and divergent if $p \leq 1$.

**(b)** For what pairs $p, q$ is the series $\sum_{2}^{\infty} n^{-p} \log^q n$ convergent?

9. Let $f$ be differentiable with continuous derivative on a finite closed interval $I$. Prove that $f$ is uniformly continuous on $I$.

**Answer:** The derivative $f'$ is bounded (continuous on a compact interval). By the mean value theorem $|f(y) - f(x)| \leq \sup |f'| |y - x|$.

10. Let $f_n$ be positive continuous functions on $[0, 1]$.

**(a)** Assume that $\sum_{1}^{\infty} f_n(x) > 1$ for all $x \in [0, 1]$. Prove that there exists an integer $N$ such that $\sum_{1}^{N} f_n(x) > 1$ for all $x \in [0, 1]$.

**Hint:** The sets $A_m = \{x: \sum_{1}^{m} f_n(x) > 1\}$ are open.

**Answer:** $\bigcup A_m = [0, 1]$ and as $[0, 1]$ is compact and $A_m$ form an open cover thereof (and $A_m$ increase with $m$), there is an $N$ such that $A_N = [0, 1]$.

**(b)** Assume that the series $\sum_{1}^{\infty} f_n(x)$ converges everywhere to a continuous function $F$.

(a) Prove that the convergence is uniform.

**Hint:** For all $\varepsilon > 0$, the sets $B_m = \{x: \sum_{1}^{m} f_n(x) > F(x) - \varepsilon\}$ are open.

**Answer:** $\bigcup B_m = [0, 1]$, and as above, the compactness of $[0, 1]$ provides an $N = N_\varepsilon$ such that $B_N = [0, 1]$.  

DECEMBER 4, 2009
11. Show by an example that the conclusion may be false without the assumption that $F$ is continuous.

**Answer:** Let $f_n(x)$ be nonnegative and continuous on $[0, 1]$, $f_n(1/2n) = 1$, $f_n(x) = 0$ if $|x - 1/2n| > 1/n^3$. Then $\sum f_n(x)$ converges everywhere since for every point $x$ there is at most one $n$ such that $f_n(x) \neq 0$. The limit $\sum f_n$ is not continuous at $x = 0$ (and only there), and the convergence is not uniform.

12. Assume that the series $\sum f_n(x)$ converges everywhere to an integrable function $F$. Prove that

$$\int_0^1 F(x)dx = \sum \int_0^1 f_n(x)dx.$$  

**Hints:** Since $F(x) > \sum f_n(x)$ for all $N$, we have $\int_0^1 F(x)dx \geq \sum \int_0^1 f_n(x)dx$. It is the reverse inequality that needs addressing. Use the fact that if $F(x) > c$ for all $x$ in some interval $[a, b]$, then for $N$ big enough, $\sum f_n(x) > c$ everywhere on $[a, b]$.

11. Assume that $f$ is monotone increasing on $[a, b]$, $f(a) = 0$, $f(b) = 1$.

a. Prove that $f$ is integrable.

**Answer:** Let $P = \{t_i\}$ be a partition of $[a, b]$, whose mesh $\max(t_{j+1} - t_j) \leq \delta$. The upper Darboux sum is $U(f, P) = \sum f(t_{j+1})(t_{j+1} - t_j)$ and the lower is $L(f, P) = \sum f(t_j)(t_{j+1} - t_j)$.

$$U(f, P) - L(f, P) = \sum (f(t_{j+1}) - f(t_j))(t_{j+1} - t_j) \leq \sum \max(f(t_{j+1}) - f(t_j)) \max(t_{j+1} - t_j) \leq \delta.$$

b. Prove that for all $c \in [a, b]$, the limits $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist, $\lim_{x \to c^-} f(x) = \liminf_{x \to c} f(x)$, and $\lim_{x \to c^+} f(x) = \limsup_{x \to c} f(x)$.

c. Prove that $\sum o_f(c) \leq 1$, the sum over all $c \in [a, b]$. (Remember that $o_f(c)$ denotes the oscillation (or jump) of $f$ at $c$.)

**Hint:** With the given $f$, how many points $c$ can there be with $o_f(c) > 1/n$?

d. Conclude that $f$ is continuous at all but countably many points in $[a, b]$.

12. Prove that a function of bounded variation on the interval $[0, 1]$ is integrable.

**Hint:** Prove that a function of bounded variation is the difference of two monotone functions.

**Answer:** A more direct proof is the observation that for every partition of $[0, 1]$ into ‘atoms’ $\{J_m\}_1^N$ the sum $\sum (\sup_{x \in J_m} f(x) - \min_{x \in J_m} f(x))$ is bounded by the total variation $V_f$ of $f$. This implies that if $P$ is a partition of mesh $\varepsilon$ then $U(f, P) - L(f, P) \leq \varepsilon V_f$. 

3 DECEMBER 4, 2009
13. Definition: A family \( \{ f_\alpha \}_{\alpha \in A} \) of functions with a common domain \( S \) is **equicontinuous at a point** \( c \in S \) if for every \( \varepsilon > 0 \) there exist \( \delta = \delta(\varepsilon, c) \) such that if \( x \in S \) and \( |x - c| < \delta \) then \( |f_\alpha(x) - f_\alpha(c)| < \varepsilon \) for all \( \alpha \in A \).

The family is **equicontinuous on** \( S \) if it is equicontinuous at every \( c \in S \); it is **uniformly equicontinuous on** \( S \) if for every \( \varepsilon > 0 \) there is \( \delta = \delta(\varepsilon) \) such that if \( x, y \in S \) and \( |x - y| < \delta \) then \( |f_\alpha(x) - f_\alpha(y)| < \varepsilon \) for all \( \alpha \in A \) (in other words, \( \delta = \delta(\varepsilon) \) can be taken common to all \( c \in S \) and \( \alpha \in A \)).

a. Assume that \( \{ f_\alpha \}_{\alpha \in A} \) is equicontinuous on a finite interval \([a, b]\). Prove that it is uniformly equicontinuous there.

b. Assume that \( f_n \), for \( n = 1, 2, \ldots \), are continuous real-valued functions on a closed finite interval \( I \), and that \( f_n \to f \) uniformly on \( I \).

Prove that the sequence \( \{ f_n \} \) is uniformly equicontinuous on \( I \).

Is the same statement true if \( I \) is not closed?

14. Let \( \{ f_n \} \) be a sequence of real-valued differentiable functions on a finite interval \([a, b]\), uniformly convergent to a function \( f \). Prove that \( \int_a^b f(x)dx = \lim_{n \to \infty} \int_a^b f_n(x)dx \).

a. Is it true that \( f \) is necessarily differentiable?

b. If you know that \( f \) is differentiable is it true that \( f' = \lim_{n \to \infty} f'_n \)?

Hint: Consider the example \( f_n(x) = n^{-1} \sin n^2 x \)

c. Prove that if the derivatives \( f'_n \) are continuous, and \( \{ f'_n \} \) is uniformly convergent on \([a, b]\), then \( f \) is differentiable and \( f' = \lim_{n \to \infty} f'_n \).

Hint: Start by integrating.

15. Assume that the radius of convergence of the series \( \sum_0^\infty a_n x^n \) is equal to 1. Prove the correct statements below and give a counter-example to the false ones:

a. The series converges uniformly in \([-1, 1]\).

b. The radius of convergence of the series \( \sum_0^\infty n a_n x^n \) is equal to 1.

c. The sum \( f(x) = \sum_0^\infty a_n x^n \) is a continuous function on \((-1, 1)\).

d. For every \( y \in (-1.1) \),
   \[ \int_0^y \sum_0^\infty a_n x^n dx = \sum_0^\infty \frac{a_n}{n+1} y^{n+1} \]

e. The sum \( f(x) = \sum a_n x^n \) is differentiable on \((-1, 1)\), and \( f'(x) = \sum n a_n x^{n-1} \).
16. a. Prove: If both series $\sum_{0}^{\infty} a_n$ and $\sum_{0}^{\infty} b_n$ converge absolutely, and $c_n = \sum_{0}^{n} a_m b_{n-m}$, then $\sum c_n$ converges absolutely, and $\sum_{0}^{\infty} c_n = (\sum_{0}^{\infty} a_n) (\sum_{0}^{\infty} b_n)$.

b. Define $e(x) = \sum_{0}^{\infty} \frac{1}{m!} x^m$. Show that for all real $x, y$ we have $e(x + y) = e(x)e(y)$.

17. Let $f \in C([a, b])$. Define $F(x) = \int_{a}^{x} f(t) \, dt$, for $a \leq x \leq b$.

a. Prove that $F \in C([a, b])$.

b. Prove that $F$ is in fact differentiable, and find its derivative.

18. The oscillation of a function $f$ at a point $c$ in its domain is defined by

\[(18.1) \quad o_f(c) = \lim_{x \to c} \sup_{|x-c|} f(x) - \lim_{x \to c} \inf_{|x-c|} f(x).\]

Another way to define the oscillation is: $o_f(c) = \sup A$, the supremum over all the real numbers $A$ such that for all $\delta > 0$, there exist points $x, y \in (c - \delta, c + \delta)$ with $|f(x) - f(y)| > A$.

a. Prove that the two definitions agree.

Hint: $\limsup_{x \to c} f(x) = \lim_{\delta \to 0} \sup_{|x-c| \leq \delta} f(x)$, $\liminf_{x \to c} f(x) = \lim_{\delta \to 0} \inf_{|x-c| \leq \delta} f(x)$

Answer: Denote the value of the oscillation according to the second definition by $o_f^*(c)$, keeping the notation $o_f(c)$ for the first. Denote $\limsup_{x \to c} f(x) = M$, $\liminf_{x \to c} f(x) = m$, so that $o_f(c) = M - m$. We need to show that $o_f^*(c) = M - m$.

For any $\epsilon > 0$ and $\delta > 0$, there exist points $x$ such that $|x - c| < \delta$ and $f(x) > M - \epsilon$ as well as points $y$ such that $|y - c| < \delta$ and $f(y) < m + \epsilon$. It follows that $o_f^*(c) \geq M - m$. On the other hand, for any $\epsilon > 0$ there exists $\delta_0 > 0$ such that if $\delta < \delta_0$ then $|x - c| < \delta$ implies $m - \epsilon < f(x) < M + \epsilon$. This proves that $o_f^*(f) \leq M - m$.

b. Prove that for any real-valued $f$ and any constant $B$, the set $\{c : o_f(c) \geq B\}$ is closed.

Answer: If $x_0$ is a limit point of $\{c : o_f(c) \geq B\}$ then, for any $\delta > 0$ there exist $x_1 \in \{c : o_f(c) \geq B\}$ such that $|x_0 - x_1| < \delta/2$. By the second definition of the oscillation, given $\epsilon > 0$, there exist $x, y \in (x_0 - \delta/2, x_1 + \delta/2) \cap (x_0 - \delta, x_0 + \delta)$ such that $|f(x) - f(y)| > B - \epsilon$. Thus $o_{x_0}(f) \geq B$.

c. Prove that if $f, g$ are real-valued functions, $f$ continuous and $|f(x) - g(x)| \leq \epsilon$ for all $x$, then (the oscillation) $o_g(c) \leq 2\epsilon$ for all $c$.

Answer: $|g(x) - g(y)| \leq |g(x) - f(x)| + |f(x) - f(y)| + |f(y) - g(y)| \leq 2\epsilon + |f(x) - f(y)|$. The last summand can be made arbitrarily small by imposing $|x - y| < \delta$ for sufficiently small $\delta$.

d. Prove that if $g$ is the uniform limit of a sequence $\{f_n\}$ of continuous functions then $g$ is continuous.

Answer: By part c. the oscillation of $g$ at every point is zero.

e. Give an example of a sequence of continuous functions on $[0, 1]$ which converges everywhere to a function $f$ which is not continuous at $x = 1/2$. (The convergence clearly can’t be uniform)
Answer: For example: \( f_n(x) = |x - 1/2|^{1/n} \). \( f_n(1/2) = 0 \) for all \( n \) and at every other point \( \lim f_n(x) = 1 \)

19. Prove that the series \( \sum n^{-2} \cos nt \) converges uniformly for \( \infty < t < \infty \).

20. Let \( f \) be a bounded function on \([0, 1]\). Denote by \( U \) the set of all upper Darboux sums for \( f \) (relative to all finite partitions), and by \( L \) the set of all lower Darboux sums for \( f \) (relative to all finite partitions).
   a. Prove that \( \inf U \geq \sup L \).
   b. Assume that \( f \) is integrable, that is \( \inf U = \sup L = \int_0^1 f(x) \, dx \). Prove: \( \forall \epsilon > 0, \exists \delta > 0 \) such that if \( P \) is a partition of \([0, 1]\) whose mesh size is less than \( \delta \), and \( S \) is any Riemann sum associated with \( P \), then \( |S - \int_0^1 f(x) \, dx| < \epsilon \).
   c. Assume that \( o_f(x) < \epsilon \) for some fixed \( \epsilon > 0 \) and all \( x \in [0, 1] \). Prove that \( \inf U - \sup L \leq \epsilon \).

21. Let \( f_n \) be positive continuous functions on \([0, 1]\) and assume that the series \( \sum f_n(x) \) converges everywhere to an integrable function \( F \). Prove that

\[
(21.1) \quad \int_0^1 F(x) \, dx = \sum \int_0^1 f_n(x) \, dx.
\]

Hint: Since \( F(x) > \sum f_n(x) \) for all \( N \), we have \( \int_0^1 F(x) \, dx \geq \sum \int_0^1 f_n(x) \, dx \). It is the reverse inequality that needs addressing. Prove and use the fact that if \( F(x) > c \) for all \( x \) in some interval \([a, b] \), then there exists an integer \( N \) such that \( \sum f_n(x) > c \) everywhere on \([a, b]\).

22. Let \( f_n \) be positive continuous functions on \([0, 1]\) and assume that the series \( \sum f_n(x) \) converges everywhere to a continuous function \( F \). Prove that the convergence is uniform.

Hint: Use the compactness of \([0, 1]\) and the fact that if \( g \) is continuous, then for every real \( \lambda \), the set \( \{ x : g(x) > \lambda \} \) is open.

23. Assume \( f \in C^n([-1, 1]) \), and \( \sup_{|x| \leq 1} |f^{(n)}(x)| \leq 2^n \).
   a. Prove that \( f = \sum f^{(j)}(0) x^j \).
   b. What is the radius of convergence of the series \( \sum f^{(j)}(0) x^j \)?

24. Let \( f \) be infinitely differentiable on \([-1, 1] \), and assume that \( f^{(n)}(0) = 0 \) for all \( n \geq 0 \). Assume also that there exists an infinite sequence \( n_k \) such that \( |f^{(n_k)}(x)| \leq 3 \) for all \( x \in [-1, 1] \). Prove that \( f = 0 \) identically.

Hint: Taylor’s theorem (with remainder).

25. A sequence \( \{f_n\} \) (of real-valued functions on an interval \( I \)) is uniformly Cauchy if for all \( \epsilon > 0 \) there exists \( N = N(\epsilon) \) such that if \( n, m > N \) then \( |f_n(x) - f_m(x)| \leq \epsilon \) for all \( x \in I \).
Prove that a sequence \( \{f_n\} \) is uniformly Cauchy if, and only if, there exists a function \( f \) such that \( f_n \to f \) uniformly.

**Answer:** If \( f_n \to f \) uniformly then for every \( \varepsilon > 0 \) there exist \( N \) such that if \( n > N \) then \( |f_n(x) - f(x)| \leq \varepsilon/2 \) for all \( x \in I \). If both \( n > N \) and \( m > N \), then \( |f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \leq \varepsilon \) for all \( x \in I \).

Conversely, if \( \{f_n\} \) is uniformly Cauchy, it is Cauchy at every point and therefore converges pointwise to a well defined \( f \). We need to show that the convergence is uniform. Let \( \varepsilon > 0 \) and let \( N = N(\varepsilon) \) be such that if \( m, n > N \) then \( |f_n(x) - f_m(x)| \leq \varepsilon \) for all \( x \in I \). We have \( |f(x) - f_n(x)| = \lim_{m \to \infty} |f_m(x) - f_n(x)| \leq \varepsilon \) for all \( x \) as soon as \( n > N(\varepsilon) \).

26. Let \( f \) be integrable on \([0, 1]\) and let \( \varepsilon > 0 \). Prove that there is a continuous function \( g \) on \([0, 1]\) such that \( g(x) \leq f(x) \) for all \( x \), and

\[
\varepsilon + \int_0^1 g(x)dx > \int_0^1 f(x)dx
\]

27. Assume that \( f \) is differentiable in \([a, b]\), \( f'(a) > 0 \), and \( f'(b) < 0 \). Prove that \( \max_{a \leq x \leq b} f'(x) \) is attained at a point \( c \) in \((a, b)\), and that \( f'(c) = 0 \). Extend this and show that derivatives of differentiable functions on an interval \( I \) have the intermediate value property: if \( c, d \in I \) then every value between \( f'(c) \) and \( f'(d) \), is assumed by \( f' \) somewhere between \( c \) and \( d \).

28. Let \( f \) be differentiable on \( \mathbb{R} \), and assume that \( \sup_{x \in \mathbb{R}} |f'(x)| = a < 1 \).

a. Select \( s_0 \in \mathbb{R} \) and define (recursively) \( s_{n+1} = f(s_n) \).

Prove that \( \{s_n\} \) is a convergent sequence.

b. Prove that \( s_\infty = \lim_{n \to \infty} s_n \) is a fixed point of \( f \), that is: \( f(s_\infty) = s_\infty \).

c. Prove that \( s_\infty \) is independent of the choice of \( s_0 \).

Hint: Show that there is only one fixed point for \( f \).

29. a. Let \( f \in C([a, b]) \) be differentiable. Assume that \( f(x_0) = f(x_1) = 0 \), \( a \leq x_0 < x_1 \leq b \). Prove that there exists \( y \in [x_0, x_1] \) such that \( f'(y) = 0 \).

b. Assume that \( f \in C^5([a, b]) \), that is: the successive derivatives \( f^{(j)} \) exist and are continuous for \( j = 1, \ldots, 5 \), and that \( f \) has at least 15 distinct zeros in the interval. Prove that the fifth derivative \( f^{(5)} \) has at least 10 zeros in the interval.

30. Let \( E \subset [0, 1] \) be infinite, and denote by \( L \) the set of all its limit points.

a. Prove that the set \( L \) is closed.

b. Prove that \( \sup L = \lim \sup E \).

**Answer:** Let \( y \) be a limit point of \( L \) and let \( \varepsilon > 0 \). The set \( L \cap (y - \varepsilon/2, y + \varepsilon/2) \) is non-empty (in fact, infinite). Let \( z \in L \cap (y - \varepsilon/2, y + \varepsilon/2) \).

The set \( E \cap (z - \varepsilon/2, z + \varepsilon/2) \) is infinite, and is contained in \( E \cap (y - \varepsilon, y + \varepsilon) \).
We have shown that $E \cap (y - \varepsilon, y + \varepsilon)$ is infinite for every $\varepsilon > 0$ so that $y \in L$. Thus, limit points of $L$ are in $L$, and $L$ is closed.

31. Let $(X, \rho)$ be a metric space. The sets below are assumed to be subsets of $X$.
   
a. Prove that an arbitrary union of open sets is open.
   
b. Prove that a finite intersection of open sets is open.
   
c. Prove that a finite union of closed sets is closed.
   
d. Prove that an arbitrary intersection of closed sets is closed.

   Let $E$ be a set and let $\{E_\alpha : \alpha \in \mathcal{A}\}$ be the collection of all the closed sets containing $E$. Then $\overline{E} = \cap E_\alpha$ is a closed set, containing $E$ and in fact the smallest closed set containing $E$. It is called the closure of $E$.

   e. Prove that $\overline{E}$ is the union of $E$ with the set of all its limit points.

   Let $E$ be a set and let $\{O_\alpha : \alpha \in \mathcal{A}\}$ be the collection of all the open sets contained in $E$. Then $E^o = \cup O_\alpha$ is an open set, contained in $E$ and in fact the biggest open set contained in $E$. It is called the interior of $E$.

32. Prove that a compact metric space is totally bounded and complete. Prove also that a totally bounded complete metric space is compact.

33. Let $\{a_n\}$ be a monotone decreasing sequence of positive numbers and $\lim_{n \to \infty} a_n = 0$. Prove that $\sum_{1}^{\infty} (-1)^{n+1} a_n$ converges and show that its sum $S$ satisfies $a_1 \geq S \geq a_1 - a_2$.

   **Answer:** Write $S_m = \sum_{1}^{m} (-1)^n a_n$. Then $S_{2m} - S_{2m-2} = a_{2m-1} - a_{2m} \geq 0$; in other words, the partial sums of even order form a monotone increasing sequence. Similarly, the partial sums of odd order form a monotone decreasing sequence. If $k, l$ are arbitrary integers, take $r$ bigger than either $k$ or $l$ and notice that

   $$S_{2k} \leq S_{2r} \leq S_{2r+1} \leq S_{2l+1}. $$

   Thus any partial sum of odd order is an upper bound for the partial sums of even order, and any partial sum of even order is a lower bound for the partial sums of odd order. It follows that both $S^* = \lim S_{2k+1}$ and $S^{**} = \lim S_{2k}$ exist, (bounded monotone sequences converge), and $S_{2n+1} \geq S^* \geq S^{**} \geq S_{2n}$ for every $n$. Let $\varepsilon > 0$ and let $n$ be such that $a_n \leq \varepsilon$. Then $S^n - S^{**} \leq S_{2n+1} - S_{2n} \leq \varepsilon$ and we have $S^{**} = S^* = S$. We can now rewrite the inequality above as $S_{2m+1} \geq S \geq S_{2n}$; in particular $S_1 = a_1 \geq S \geq S_2 = a_1 - a_2$.

34. Assume $a_n \geq 0$ for all $n$ and $\sum a_n < \infty$. Prove that $\sum \frac{\sqrt{a_n}}{n} < \infty$.

   **Answer:** By the Cauchy-Schwarz inequality, for any $N \in \mathbb{N}$ we have

   $$\sum_{1}^{N} \frac{\sqrt{a_n}}{n} \leq \left( \sum_{1}^{N} \frac{1}{n^2} \right)^{1/2} \left( \sum_{1}^{N} a_n \right)^{1/2} < \left( \sum_{1}^{\infty} \frac{1}{n^2} \right)^{1/2} \left( \sum_{1}^{\infty} a_n \right)^{1/2}. $$

   Since the partial sums form a bounded monotone sequence, the limit exists. Notice that the limit is bounded by the bound on the partial sums:

   $$\sum_{1}^{\infty} \frac{\sqrt{a_n}}{n} \leq \left( \sum_{1}^{\infty} \frac{1}{n^2} \right)^{1/2} \left( \sum_{1}^{\infty} a_n \right)^{1/2}. $$
35. Define \( s_n = \sum_{0}^{n} \frac{1}{j!} \), (remember that, by definition, \( 0! = 1 \)), and \( t_n = (1 + \frac{1}{n})^n \). Prove the following statements

a. Prove that \( \sum_{0}^{\infty} \frac{1}{j!} = \lim_{n \to \infty} s_n \) exists and is < 3. (The sum is denoted \( e \), it is “the base of the natural logarithm”.)

**Answer:** Since \( j! \geq 2^{j-1} \) for \( j > 1 \) we have \( s_n \leq 2 + \sum_{1}^{n-1} 2^{-j} = 3 - 2^{1-n} \).

b. For all positive \( n, t_n \leq s_n \), yet, for every positive \( m \) there exists \( N(m) \) such that if \( n > N(m) \) then \( t_n \geq s_m \).

**Answer:** By the binomial expansion \( t_n = \sum_{0}^{n} \binom{n}{j} n^{-j} \). The first two terms are (each) equal to 1, for \( j > 1 \) we have \( \binom{n}{j} n^{-j} = \frac{n(n-1)\cdots(n-j+1)}{n^j} < \frac{1}{j!} \), and hence \( t_n \leq s_n \). On the other hand, for any fixed \( j \), and uniformly for \( j \leq m \), \( \binom{n}{j} n^{-j} \to 1 \) as \( n \to \infty \). It follows that for any \( \varepsilon > 0 \), the sum of the first \( m+1 \) terms in the sum expressing \( t_n \) exceeds \( s_m - \varepsilon \) when \( n \) is sufficiently large.

Take \( \varepsilon = \frac{1}{(2m)!} \) (small compared to the \((m+2)\) nd summand for \( t_n \)) and, for sufficiently large \( n \), we have \( t_n \geq s_m - \varepsilon + \frac{1}{2(2m+2)!} > s_m \).

c. \( \lim_{n \to \infty} (1 + \frac{1}{n})^n = e \).

**Answer:** By a. we have \( \lim_{n \to \infty} t_n \leq \lim_{n \to \infty} s_n = e \).

By b. we have \( \lim_{n \to \infty} t_n \geq \lim_{m \to \infty} s_m = e \).

36. Let \( (X, \rho) \) and \( (Y, d) \) be metric spaces. Prove that a map \( f : X \to Y \) is continuous if and only if the inverse image \( f^{-1}(O) = \{x : f(x) \in O \} \) is open in \( X \) for every open set \( O \subset Y \).

37. Let \( f \) and \( g \) be continuous real-valued functions on \([0, 1]\).

Prove that the set \( \{(f(x), g(x)) : x \in [0.1] \} \) in \( \mathbb{R}^2 \) is connected and compact.

38. Let \( f \) be real valued on \( I = [0, 1] \);

a. Assume that \( o_f(x) \leq a \) for all \( x \in I \). Prove that for every \( a' > a \) there exists a \( \delta > 0 \) such that if \( x, y \in I \) and \( |x - y| < \delta \) then \( |f(x) - f(y)| < a' \).

Observe that, in the context above, \( U(f,P) - L(f,P) \leq a' \) for all partitions of mesh \( < \delta \).

b. Denote \( J(f, \varepsilon) = \{x : o_f(x) \geq \varepsilon \} \) and let \( C(f, \varepsilon, N) \) be the number of intervals of the form \([j/N, (j+1)/N], j \in \mathbb{N} \), that have a nonempty intersection with \( J(f, \varepsilon) \).

Prove that \( f \) is integrable if, and only if, \( \lim_{N \to \infty} \frac{C(f, \varepsilon, N)}{N} = 0 \) for all \( \varepsilon > 0 \).

39. Define \( l(x) = \int_{1}^{x} \frac{dy}{y} \).

a. Check that \( l(x) \) is well defined for all \( x > 0 \).

b. \( l(x) \) is strictly increasing, differentiable, tending to \( -\infty \) as \( x \to 0 \) and to \( +\infty \) as \( x \to \infty \).

c. \( l(ab) = l(a) + l(b) \) for all positive \( a, b \). In particular: \( l(1/e) = -l(x) \).
d. The inverse function of \( f \), denoted \( e(x) \), satisfies the functional equation \( e(a + b) = e(a)e(b) \).

e. \( e \) is differentiable and satisfies: \( e'(x) = e(x) \).

f. Prove that \( e(x) \), as defined here, satisfies: \( e(x) = \sum \frac{x^n}{n!} \).

40. Prove that an (everywhere) differentiable function \( f \) on a closed interval \([a, b]\) is bounded.

41. A closed set \( E \subset [0, 1] \) is non-dense (or nowhere dense) if its interior is empty, (that is: \( E \) contains no nonempty open set.) An arbitrary set \( E \subset [0, 1] \) is non-dense if its closure is non-dense (equivalently: if the interior of its complement is dense in \([0, 1]\)).

Prove that if \( E_n \) is non-dense for \( n = 1, 2, \ldots \), then \( \bigcup E_n \neq [0, 1] \).

42. Let \( (s_n) \) be a sequence of real numbers. Define: \( \sigma_n = \frac{1}{n} \sum_{j=1}^{n} s_j \) (for \( n = 1, 2, \ldots \)).

a. Prove

\[
\lim \inf s_n \leq \lim \inf \sigma_n \leq \lim \sup \sigma_n \leq \lim \sup s_n.
\]

Answer: Given \( \varepsilon > 0 \), there exists \( N \) such that for \( n > N \)

\[
\lim \inf s_j - \varepsilon \leq s_n \leq \lim \sup s_j + \varepsilon.
\]

It follows that, for all \( m \),

\[
\frac{N}{m} \inf s_j + \frac{m-N}{m} \lim \inf s_j - \varepsilon \leq \frac{1}{m} \sum_{j=1}^{m} s_j \leq \frac{N}{m} \sup s_j + \frac{m-N}{m} \lim \sup s_j + \varepsilon.
\]

As \( m \to \infty \), \( \frac{N}{m} \to 0 \) and \( \frac{m-N}{m} \to 1 \); we obtain

\[
\lim \inf s_n - \varepsilon \leq \lim \inf \sigma_n \leq \lim \sup \sigma_n \leq \lim \sup s_n + \varepsilon.
\]

Since \( \varepsilon \) is arbitrary we obtain the desired inequality.

b. Prove that if \( \lim_{n \to \infty} s_n = a \) then \( \lim_{n \to \infty} \sigma_n = a \). Give an example of a sequence \( \{s_n\} \) which is not convergent but for which \( \lim \sigma_n \) exists.

Answer: If \( \lim \inf s_n = \lim \sup s_n = a \) then all the inequalities in (??) are in fact equalities and \( \lim \sigma_n = a \).

If \( a_n = (-1)^n \) then \( s_{2n} = 0 \), \( s_{2n+1} = -1 \) and \( \lim \sigma_n = -1/2 \).

43. For \( n \in \mathbb{N} \) let \( a_n \) be an integer in \([0, 1, \ldots, 9]\). Prove that the infinite decimal expansion, \( 0.a_1a_2a_3 \ldots \) represents a unique real number in the following sense: the sequence \( x_n = \sum_{j=1}^{n} a_j 10^{-j} \) converges to a limit (denoted \( x = \sum_{j=1}^{\infty} a_j 10^{-j} \)). Conversely, prove that every number \( x \in (0, 1) \) has such representation(s).

44. Construction of a continuous nowhere differentiable function: Let \( a_n = 10^{-n} \). Observe that for every \( m \), \( \sum_{m+1}^{\infty} a_n < \frac{1}{8} a_m \). Let \( \lambda_n = (10n)! \) and define \( F \) by \( F(t) = \sum_{m=1}^{n} a_m \cos \lambda_m t \).
**Claim:** *F* is nowhere differentiable.

a. Write \( \Phi_m(t) = \sum_{n=1}^{m-1} a_n \cos \lambda_n t \) and verify that \( |\Phi'_m(t)| < \sum_{n<m} a_n \lambda_n < \lambda_{m-1} \).

Hence, for \( h \) such that \(|h| < 2\pi \lambda_{m-1}^{-1} \),

\[
(44.1) \quad \left| \Phi_m(t+h) - \Phi_m(t) \right| \leq 2\pi \lambda_{m-1} \lambda_{m-1}^{-1}
\]

b. Write \( \Psi_m(t) = \sum_{n=m+1}^{\infty} a_n \cos \lambda_n t \) and observe that \( |\Psi_m(t)| < \frac{1}{8} a_m \).

c. **Proof of the claim:** Let \( t \in \mathbb{R} \) be arbitrary; we show that \( F \) is not differentiable at \( t \).

If \( F(t) \geq 0 \) let \( x_m < t \) be the be the biggest such that \( \cos \lambda_m(t) = -1 \).

Observe that \( 0 < t - x_m < 2\pi \lambda_{m-1}^{-1} \) so that

\[
F(t) - F(x_m) = \left( \Phi_m(t) - \Phi_m(x_m) \right) + \left( \Psi_m(t) - \Psi_m(x_m) \right) + \left( a_m \cos \lambda_m t - a_m \cos \lambda_m x_m \right)
\]

and since \( \cos \lambda_m x_m = -1 \) we have \( F(t) - F(x_m) \geq a_m - 2\pi \lambda_{m-1} \lambda_{m-1}^{-1} - a_m / 4 > a_m / 2 \) and

\[
(44.2) \quad \frac{F(t) - F(x_m)}{t - x_m} > \frac{a_m \lambda_m}{4\pi}
\]

Which goes to \(+\infty\) with \( m \).

If \( F(t) \leq 0 \) we take \( x_m > t \) be the smallest such that \( \cos \lambda_m(t) = 1 \).

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**45.** A construction of an (everywhere) differentiable function that is nowhere monotone—its derivative takes both positive and negative values in every interval (and hence is not integrable).

Define \( \varphi(x) = \frac{1}{\sqrt{1+|x|}} \).

a. Prove that for all pairs \( a, b \in \mathbb{R}, \ a < b \), we have

\[
(45.1) \quad \frac{1}{b-a} \int_a^b \varphi(x) dx \leq 4 \min_{a \leq x \leq b} \varphi(x).
\]

**Hint:**

(a) Show that one may assume that \( b > |a| \).

(b) Show that if \( a > 0 \), \( \frac{1}{b-a} \int_a^b \varphi(x) dx \leq \frac{1}{b} \int_0^b \varphi(x) dx \), while if \(-b \leq a < 0\) then

\[
\frac{1}{b-a} \int_a^b \varphi(x) dx \leq \frac{2}{b} \int_0^b \varphi(x) dx.
\]

(c) Notice that \( m(b) = \min_{0 \leq x \leq b} \varphi(x) = \varphi(b) = \frac{1}{\sqrt{1+b}} \) while the average

\[
A(b) = \frac{1}{b} \int_0^b \varphi(x) dx = \frac{1}{2}(\sqrt{b} - b - 1).
\]

Show that the ratio \( A(b)/m(b) \) is monotone increasing, and converges to 2 as \( b \to \infty \).

b. Prove that inequality (45.1) remains valid if one replaces \( \varphi(x) \) by \( \varphi(\lambda(x-x_0)) \) where \( \lambda \) and \( x_0 \) are arbitrary real numbers.

c. Let \( a_n > 0, \ \lambda_n \) and \( x_n \) real numbers. Write \( \Phi_n(x) = \int_0^x \varphi(\lambda_n(y-x_n)) dy \).

Assume that the series \( \sum_n a_n \varphi(\lambda_n(x-x_n)) \) converges for some point \( x = \bar{x} \). Prove that \( \sum_n a_n \Phi_n(x) \) converges uniformly in every bounded set in \( \mathbb{R} \).

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11 DECEMBER 4, 2009
d. Prove that $F(x) = \sum_1^\infty a_n \Phi_n(x)$ is differentiable at every point $x$ at which $\sum_1^\infty a_n \varphi(\lambda_n(x - x_n))$ converges, and that $F'(x) = \sum_1^\infty a_n \varphi(\lambda_n(x - x_n))$ at these points. In particular, if the series $\sum_1^\infty a_n \varphi(\lambda_n(x - x_n))$ converges everywhere, $F$ is differentiable everywhere.

e. Notice that if $\lambda$ is large, $\varphi(\lambda(x - x_0))$ has a very sharp spike near $x_0$ and is quite small away from $x_0$. Use this to construct a function $F$ which is differentiable everywhere, whose derivative is bounded, but is not integrable.

Hint: Show that for an appropriate choice of $\{a_n\}, a_n > 0, \{\lambda_n\}$ and $\{x_n\}$, the sum $f = \sum_1^\infty a_n \varphi(\lambda_n(x - x_n))$ satisfies:

(a) $f(0) < 1/2,$
(b) For any interval $I \subset [0, 1]$, we have $1 < \sup_{x \in I} f(x) < 2.$

Remark: Given two disjoint sequences $\{q_n\}$ and $\{r_n\}$, one can choose $\{\lambda_n\}, \{x_n\}$ and $\{a_n\}$ so that

$$\sum_1^\infty a_n \varphi(\lambda_n(x - x_n)) \begin{cases} 1 & \text{for } x \in \{q_n\} \\ < 1 & \text{for } x \in \{r_n\} \end{cases}.$$ 

Reversing the role of $\{q_n\}$ and $\{r_n\}$, repeating, and taking the difference, one can obtain an example of an everywhere differentiable function whose derivative is positive on $\{q_n\}$ and negative on $\{r_n\}$, even if both sequences are dense!