Math 115, midterm topics and sample problems

1. Prove that if \(a, b \in \mathbb{Z}\) and \(n \in \mathbb{N}\), then \(a - b\) divides \(a^n - b^n\) (in \(\mathbb{Z}\)).

   **Answer:** By induction: clear for \(n = 1\) and \(n = 2\). Verify and use the equation
   \[a^{n+1} - b^{n+1} = (a + b)(a^n - b^n) - ab(a^{n-1} - b^{n-1})\]

2. Prove that for all \(n \in \mathbb{N}\),
   \[\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2\]

3. Prove that the completeness axiom for \(\mathbb{R}\) as stated in §4.4, namely:
   
   a. *Every nonempty subset \(S\) of \(\mathbb{R}\) that is bounded above has a least upper bound in \(\mathbb{R}\)*, (denoted \(\sup S\) or \(\text{lub} S\)).

   is equivalent to the statement
   
   b. *Every Cauchy sequence of real numbers has a limit in \(\mathbb{R}\).*

   **Answer:** We need to show that each statement implies the other

   \(a. \implies b.\) Assume the completeness axiom.

   **Preparation:** Observe that for a bounded monotone non-decreasing sequence \(y_n\) one has
   \[\lim y_n = \sup\{y_n\}\]
   (The sup exists by the Completeness axiom. By the definition of sup there exist, for any \(\varepsilon > 0\), values of \(N\) such that \(y_N > \sup\{y_n\} - \varepsilon\) and if \(n > N\) we have \(y_N \leq y_n \leq \sup\{y_n\}\).

   Similarly for a bounded monotone non-increasing sequence \(y_n\) one has \(\lim y_n = \inf\{y_n\}\). The completeness axiom now implies that every bounded monotone sequence \(y_n\) of real numbers has a (real number) limit.

   This implies the existence, for any bounded sequence \(\{x_n\}\), of the limits
   \[\limsup x_n = \lim_{m \to \infty} \sup_{n > m} x_n \quad \text{and} \quad \liminf x_n = \lim_{m \to \infty} \inf_{n > m} x_n.\]
   (since \(y_m = \sup_{n > m} x_n\) and \(z_m = \inf_{n > m} x_n\) are monotone and bounded.)

   Let \(\{x_n\} \subseteq \mathbb{R}\) be a Cauchy sequence; it is clearly bounded. Write \(S = \limsup x_n, s = \liminf x_n\) then \(S \geq s\) and we claim that in fact \(S = s\) so that \(\lim x_n = S\) and \(\{x_n\}\) has a limit in \(\mathbb{R}\). We show that \(S = s\) by showing that the opposite assumption in not consistent with the Cauchy condition on \(\{x_n\}\). Assume \(S - s > 0\) and set \(\varepsilon = \frac{s - S}{2}\); there are arbitrarily large values \(n, m\) such that \(x_n < s + \varepsilon\) and \(x_m > S - \varepsilon\) so that \(x_m - x_n > \varepsilon\), which contradicts the assumption that \(\{x_n\}\) is a Cauchy sequence.

   \(b. \implies a.\) Assume “Every Cauchy sequence of real numbers has a limit in \(\mathbb{R}\)”.

   Let \(A\) be a set which is bounded above, and let \(x_1\) be an upper bound. If \(x_1 = \sup A\) we are done. Otherwise let \(k\) be the smallest integer such that \(x_1 - \frac{1}{k}\) is an upper bound (for \(A\)) and set \(x_2 = x_1 - \frac{1}{k}\). Repeat, defining \(x_{n+1} = x_n - \frac{1}{k_n}\) where \(k_n\) is the smallest integer such that \(x_n - \frac{1}{k_n}\) is an upper bound for \(A\); (if at any stage \(x_n = \sup A\) we are done). The sequence \(\{x_n\}\) is bounded below (by \(A\)) so that \(\sum k_n^{-1} < \infty\).
4. Prove that \( \mathbb{R}^3 \) with the metric \( d(x,y) = \sqrt{\sum_{i=1}^{3}(x_i - y_i)^2} \) (where \( x = (x_1, x_2, x_3) \) and \( y = (y_1, y_2, y_3) \)) is complete: every Cauchy sequence in \( \mathbb{R}^3 \) has a limit.

**Answer:** The sequence \( \{w_n\}, w_n = (x_n, y_n, z_n) \), is a Cauchy sequence in \( \mathbb{R}^3 \) if and only if each of the sequences \( \{x_n\} \{y_n\} \{z_n\} \) is a Cauchy sequence in \( \mathbb{R} \). Thus, if \( \{w_n\} \) is a Cauchy sequence then the limits \( X = \lim_{n\to\infty} x_n, Y = \lim_{n\to\infty} y_n, \) and \( Z = \lim_{n\to\infty} z_n \) exist and since all three coordinates in \( (X,Y,Z) - w_n \) converge to zero we have

\[
(X,Y,Z) = \lim_{n\to\infty} w_n.
\]

**Notation and terminology:** Let \( E \subset \mathbb{R} \),

- \( x \in E \) is an isolated point of \( E \) if there exists \( \varepsilon > 0 \) such that \( (x - \varepsilon, x + \varepsilon) \cap E = \{x\} \).
- \( x \in E \) is a limit point of \( E \) if for every \( \varepsilon > 0 \) the set \( (x - \varepsilon, x + \varepsilon) \cap E \) is infinite.

The set \( E \) is closed if it contains all its limit points.

A set \( E \) is open if its complement is closed.

5. Prove that a set \( E \) is open if and only if for all \( x \in E \) there exists \( \varepsilon > 0 \) such that \( (x - \varepsilon, x + \varepsilon) \subset E \).

Give an example of a set which is neither open nor closed.

6. Show that \( x \in E \) is an limit point of \( E \) if for every \( \varepsilon > 0 \) the set \( (x - \varepsilon, x + \varepsilon) \cap E \) contains at least one point different from \( x \).

7. Let \( E \) be a bounded closed set on \( \mathbb{R} \). Prove that \( \sup E \in E \). In other words: \( \sup E = \max E \).

**Hint:** The complement of a closed set is open.

8. Let \( E \subset [0, 1] \) be infinite. Prove that the set \( L \) of all the limit points of \( E \) is non-empty and closed.

**Answer:** Let \( y \) be a limit point of \( L \) and let \( \varepsilon > 0 \). The set \( L \cap (y - \varepsilon/2, y + \varepsilon/2) \) is not empty. Let \( z \in L \cap (y - \varepsilon/2, y + \varepsilon/2) \). The set \( E \cap (z - \varepsilon/2, z + \varepsilon/2) \) is not empty. Let \( x \in E \cap (z - \varepsilon/2, z + \varepsilon/2) \); we have \( |x - y| < \varepsilon \). This shows that \( E \cap (y - \varepsilon, y + \varepsilon) \) is not empty for every \( \varepsilon > 0 \), that is \( y \) is a limit point of \( E \). In other words \( y \in L \) and we have shown that limit points of \( L \) are in \( L \), so \( L \) is closed.

9. a. Given a bounded sequence \( \{a_n\} \) of real numbers.

Prove that \( A = \limsup a_n \) (defined by: \( \limsup a_n = \lim_{N \to \infty} \sup_{n \geq N} a_n \)) is the unique real number that has the property:

**P:** For every \( \varepsilon > 0 \), the set \( \{n : a_n > A + \varepsilon\} \) is finite while the set \( \{n : a_n < A - \varepsilon\} \) is infinite.

**Answer:** Since, for every \( \varepsilon > 0, A + \varepsilon > \lim_{N \to \infty} \sup_{n \geq N} a_n \), there exists \( M \) such that \( A + \varepsilon > \sup_{n \geq M} a_n \). That means that only elements with index \( < M \) can exceed \( A + \varepsilon \).

On the other hand, \( A - \varepsilon < \lim_{N \to \infty} \sup_{n \geq N} a_n \) and since \( \{b_N = \sup_{n \geq N} a_n\} \) is non-increasing (as a sequence indexed by \( N \)) we have \( A - \varepsilon < \sup_{n \geq N} a_n \) for every \( N \), and there can be no last index in the set \( \{a_n : a_n > A - \varepsilon\} \).
10. For every \( a_n \geq A + \varepsilon \) then \( B - \varepsilon = A + \varepsilon \), so that the set \( \{ n : a_n > A + \varepsilon \} = \{ n : a_n > B - \varepsilon \} \) is finite, and \( B \) does not have property \( \mathcal{P} \). If \( B < A \) and \( B \) has property \( \mathcal{P} \) then the argument given shows that \( A \) cannot have it.

b. For an infinite bounded set \( E \subset \mathbb{R} \) define: \( \limsup E = \sup \{ x : x \text{ is a limit point of } E \} \). Prove that \( A = \lim sup E \) if and only if it has the analog of property \( \mathcal{P} \), namely: for every \( \varepsilon > 0 \), the set \( E \cap [A + \varepsilon, +\infty) \) is finite, while \( E \cap [A - \varepsilon, +\infty) \) is infinite.

For \( n \in \mathbb{N} \) let \( a_n \) be an integer in \([0,1,\ldots,9]\). Prove that the infinite decimal expansion, \( 0.a_1a_2a_3 \ldots \), represents a unique real number \( \alpha \) in the following sense: the sequence \( x_n = \sum_{j=1}^{n} a_j 10^{-j} \) converges to a limit (denoted \( \alpha = \sum_{j=1}^{\infty} a_j 10^{-j} \)). Conversely, prove that every number \( \alpha \in (0,1) \) has such representation(s).

**Answer:**

a. For every choice of the coefficients \( a_j \), the sequence \( x_n = \sum_{j=1}^{n} a_j 10^{-j} \) is a Cauchy sequence, hence convergent. The Cauchy condition is satisfied since for \( m > n \),

\[
|x_m - x_n| = \sum_{n+1}^{m} a_j 10^{-j} \leq 9 \cdot 10^{-n},
\]

which goes to zero as \( n \to \infty \).

b. Given \( x \in (0,1) \) we define the coefficients \( a_n \) inductively: \( a_1 = \lfloor 10x \rfloor \), \( \lfloor y \rfloor \) denotes the integer part of \( y \), that is the biggest integer smaller than or equal to \( y \); check that \( 0 \leq x - a_1 10^{-1} < 10^{-1} \). Assume that \( a_j \) has been defined for \( j \leq n \) and that \( 0 \leq x - x_n < 10^{-n} \) where \( x_n = \sum_{j=1}^{n} a_j 10^{-j} \); then set \( a_{n+1} = \lfloor 10^{n+1} (x - x_n) \rfloor \).

c. What numbers have terminating decimal expansions and which have ultimately periodic ones?

**Answer:** Numbers with terminating decimal expansions are clearly rationals that can be written as \( p10^{-n} \) with \( p \) and \( n \) integers. Rational numbers and only rational numbers have ultimately periodic expansions (see Theorem 16.5, page 84, in the textbook).

11. Prove or disprove (by giving a counter-example) each of the following statements: (the sequences \{\( a_n \}\} and \{\( b_n \}\} are assumed to be bounded sequences of real numbers)

a. \( \limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n \).

b. \( \limsup a_n b_n \leq \limsup a_n \limsup b_n \).

c. \( \liminf(a_n + b_n) \leq \liminf a_n + \liminf b_n \).

d. \( \liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n \).

**Answer:**

a. True: For every \( \varepsilon > 0 \), appropriate \( N \) and \( j > N \), \( a_j < \limsup a_n + \varepsilon \), \( b_j < \limsup b_n + \varepsilon \) so that \( a_j + b_j < \limsup a_n + \limsup b_n + 2\varepsilon \).
b. False (unless, e.g., both sequences are positive): \( a_n = 1 + (-1)^n \) and \( b_n = -1 \). Then \( \limsup a_n = 2, \limsup b_n = -1 \) so their product is \(-2\) while \( \limsup a_n b_n = 0 \). of the limits

c. False: \( a_n = 1 + (-1)^n, \ b_n = 1 - (-1)^n \). We have \( \liminf a_n = \liminf b_n = 0 \), while \( \liminf(a_n + b_n) = 1 \).

d. True. This is a. in which everything is multiplied by \(-1\).

12. Let \( f \) be continuous on \([0, 1]\). Prove:
   a. \( f \) is bounded on \([0, 1]\).
   b. \( f \) is uniformly continuous on \([0, 1]\).

13. Let \( (s_n) \) be a sequence of real numbers. Define: \( \sigma_n = \frac{1}{n} \sum_{j=1}^{n} s_j \) (for \( n = 1, 2, \ldots \)).
   a. Prove: \( \liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n \).
   
   **Answer:** Given \( \varepsilon > 0 \), there exists \( N \) such that for \( n > N \)
   \[ \liminf s_j - \varepsilon \leq s_n \leq \limsup s_j + \varepsilon. \]

   It follows that, for all \( m \),
   \[ \frac{N}{m} \inf s_j + \frac{m-N}{m} \liminf s_j - \varepsilon \leq \frac{1}{m} \sum_{j=1}^{m} s_j \leq \frac{N}{m} \sup s_j + \frac{m-N}{m} \limsup s_j + \varepsilon. \]

   As \( m \to \infty \), \( \frac{N}{m} \to 0 \) and \( \frac{m-N}{m} \to 1 \); we obtain
   \[ \liminf s_n - \varepsilon \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n + \varepsilon. \]

   Since \( \varepsilon \) is arbitrary we obtain the desired inequality.

   b. Prove that if \( \lim_{n \to \infty} s_n = a \) then \( \lim_{n \to \infty} \sigma_n = a \).
   
   **Answer:** If \( \liminf s_n = \limsup s_n = a \) then all the inequalities in (??) are in fact equalities and \( \lim \sigma_n = a \).

   c. Give an example of a sequence \( \{s_n\} \) which is not convergent but for which \( \lim \sigma_n \) exists.
   
   **Answer:** If \( a_n = (-1)^n \) then \( s_{2n} = 0, s_{2n+1} = -1 \) and \( \lim \sigma_n = -1/2 \).

14. Let \( \{a_n\} \) be a monotone decreasing sequence of positive numbers such that \( \lim_{n \to \infty} a_n = 0 \).
   Prove that \( \sum_{n=1}^{\infty} (-1)^{n+1} a_n \) converges and show that its sum \( S \) satisfies \( a_1 \geq S \geq a_1 - a_2 \).
   
   **Answer:** Write \( S_m = \sum_{n=1}^{m} (-1)^{n+1} a_n \). Then \( S_{2m} - S_{2m-2} = a_{2m-1} - a_{2m} \geq 0 \); in other words, the
   partial sums of even order form a monotone increasing sequence. Similarly, the partial sums of odd order form a monotone decreasing sequence. If \( k, l \) are arbitrary integers, take \( r \) bigger than either \( k \) or \( l \) and notice that
   \[ S_{2k} \leq S_{2r} \leq S_{2r+1} \leq S_{2l+1}. \]
Thus any partial sum of odd order is an upper bound for the partial sums of even order, and any partial sum of even order is a lower bound for the partial sums of odd order. It follows that both \( S^* = \lim S_{2k+1} \) and \( S^{**} = \lim S_{2k} \) exist, (bounded monotone sequences converge), and \( S_{2n+1} \geq S^* \geq S^{**} \geq S_{2n} \) for every \( n \). Let \( \varepsilon > 0 \) and let \( n \) be such that \( a_n \leq \varepsilon \). Then \( S^* - S^{**} \leq S_{2n+1} - S_{2n} \leq \varepsilon \) and we have \( S^{**} = S^* = S \). We can now rewrite the inequality above as \( S_{2m+1} \geq S \geq S_{2n} \); in particular \( S_1 = a_1 \geq S \geq S_2 = a_1 - a_2 \).

15. Assume \( a > 0, p \) an arbitrary real number. Prove: \( \lim_{n \to \infty} n^p (1+a)^{-n} = 0 \).

16. Define \( s_n = n^p \sum_{j=1}^{n} 1/j! \). (remember that, by definition, \( 0! = 1 \)), and \( t_n = (1 + \frac{1}{n})^n \). Prove the following statements
   a. Prove that \( \sum_{n=0}^{\infty} n^n \) exists and is \( < 3 \). (The sum is standardly denoted \( e \), it is “the base of the natural logarithm”.)
   b. For all positive \( n, t_n \leq s_n \), yet, for every positive \( m \) there exists \( N(m) \) such that if \( n > N(m) \) then \( t_n = n \).
   c. \( \lim_{n \to \infty} (1 + \frac{1}{n})^n = e \).

17. a. For what values of \( p \) is the series \( \sum_{n=1}^{\infty} n^{-p} \) convergent.
   b. For what pairs \( p, q \) is the series \( \sum_{n=2}^{\infty} n^{-p} \log^q n \) convergent.

18. Assume \( a > 0, p \) an arbitrary real number. Prove: \( \lim_{n \to \infty} n^p (1+a)^{-n} = 0 \).
   **Answer:** By the binomial expansion, \( (1 + a)^n > (\frac{n}{k}) a^k \) when \( n > k \). Take \( k > p + 1 \) and remember \( \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \) so that for the constant \( C = a^{-k} 2^k! \) and \( n > 2k \) we have \( (1 + a)^n > C^{-1} n^k \) and \( n^p (1+a)^{-n} < Cn^{p-k} < \frac{C}{n} \) which converges to zero.

19. Assume \( a_n \geq 0 \) for all \( n \) and \( \sum a_n < \infty \). Prove that \( \sum \frac{a_n}{n} < \infty \).
   **Answer:** By the Cauchy-Schwarz inequality, for any \( N \in \mathbb{N} \) we have
   \[
   \sum_{n=1}^{N} \frac{\sqrt{a_n}}{n} \leq \left( \sum_{n=1}^{N} n^{-2} \right)^{1/2} \left( \sum_{n=1}^{N} a_n \right)^{1/2} < \left( \sum_{n=1}^{\infty} n^{-2} \right)^{1/2} \left( \sum_{n=1}^{\infty} a_n \right)^{1/2}.
   \]
   Since the partial sums form a bounded monotone sequence, the limit exists. Notice that the limit is bounded by the bound on the partial sums:
   \[
   \sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} \leq \left( \sum_{n=1}^{\infty} n^{-2} \right)^{1/2} \left( \sum_{n=1}^{\infty} a_n \right)^{1/2}.
   \]

20. Let \( f \) be a real-valued function defined in the interval \([a, b] \subset \mathbb{R} \). Define
   \[
   \limsup_{x \to c} f(x) = \lim_{\delta \to 0} \sup_{c - \delta \leq x \leq c + \delta} f(x) \quad \liminf_{x \to c} f(x) = \lim_{\delta \to 0} \inf_{c - \delta \leq x \leq c + \delta} f(x).
   \]
The oscillation of a function $f$ at a point $c$ in its domain is defined by

$$o_c(f) = \limsup_{x \to c} f(x) - \liminf_{x \to c} f(x).$$

Another way to define the oscillation is: $o_c(f) = \sup A$, the supremum over all the real numbers $A$ such that for all $\delta > 0$, there exist points $x, y \in (c - \delta, c + \delta)$ with $|f(x) - f(y)| > A$.

a. Prove that the two definitions agree.

**Answer:** We need to show that

$$\lim_{\delta \to 0} \sup_{x, y \in (c - \delta, c + \delta)} |f(x) - f(y)| = \limsup_{x \to c} f(x) - \liminf_{x \to c} f(x).$$

By the definition of lim sup and lim inf, for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - c| < \delta$ then

$$\liminf_{x \to c} f(x) - \varepsilon < f(x) < \limsup_{x \to c} f(x) + \varepsilon.$$

It follows that if $x, y \in (c - \delta, c + \delta)$ then $|f(x) - f(y)| < o_c(f) + 2\varepsilon$, and since $\varepsilon$ is arbitrarily small, we obtain

$$\lim_{\delta \to 0} \sup_{x, y \in (c - \delta, c + \delta)} |f(x) - f(y)| \leq \limsup_{x \to c} f(x) - \liminf_{x \to c} f(x).$$

Again by the definition of lim sup and lim inf, for any $\varepsilon > 0$ and any $\delta > 0$, there exist $\tilde{x}, \tilde{y} \in (c - \delta, c + \delta)$ with $f(\tilde{x}) > \limsup_{x \to c} f(x) - \varepsilon$ and $f(\tilde{y}) < \liminf_{x \to c} f(x) + \varepsilon$. It follows that $f(\tilde{x}) - f(\tilde{y}) > o_c(f) - 2\varepsilon$, i.e.,

$$\lim_{\delta \to 0} \sup_{x, y \in (c - \delta, c + \delta)} |f(x) - f(y)| \geq \limsup_{x \to c} f(x) - \liminf_{x \to c} f(x) - 2\varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, the two definitions agree.

b. Prove that for any real-valued $f$ and any constant $B$, the set $\{c : o_c(f) \geq B\}$ is closed.

**Answer:** If $y$ is a limit point of $\{c : o_c(f) \geq B\}$, any neighborhood $(y - \delta, y + \delta)$ contains points $c$ such that $o_c(f) \geq B$. For any $\varepsilon > 0$ and $\delta > 0$, there exist pairs $\tilde{x}, \tilde{y}$ in $(c - \delta, c + \delta) \subset (y - 2\delta, y + 2\delta)$ such that $f(\tilde{x}) - f(\tilde{y}) > o_c(f) - \varepsilon$. Since $\varepsilon > 0$ and $\delta > 0$ are arbitrary, we obtain $o_y(f) \geq B$ and $y$ belongs to the set.

c. Prove that if $f$, $g$ are real-valued functions, $g$ continuous and $|f(x) - g(x)| \leq \varepsilon$ for all $x$, then (the oscillation) $o_c(f) \leq 2\varepsilon$ for all $c$.

d. Assume that $f_n, n \in \mathbb{N}$, are continuous real-valued functions on $[0, 1]$, $f$ is a function defined on $[0, 1]$, and $\lim_{n \to \infty} \sup_{0 \leq x \leq 1} |f_n(x) - f(x)| = 0$, (this condition is what is usually referred to as “$f_n$ converge to $f$ uniformly”). Prove that $f$ is continuous everywhere on $[0, 1]$.

**Answer:** Let $c \in [0, 1]$, and let $\varepsilon > 0$ be arbitrary. Let $n$ be such that $|f(x) - f_n(x)| \leq \varepsilon$ for all $x$. By the previous part (with $g = f_n$), $o_c(f) \leq 2\varepsilon$. It follows that $o_c(f)$ is identically zero, i.e., $f$ is continuous.
e. Give an example of a sequence of continuous functions on \([0, 1]\) which converges everywhere to a function \(f\) which is not continuous at \(x = 1/2\). (The convergence clearly can’t be uniform)

**Answer:** For example: \(f_n(x) = |x - 1/2|^{1/n}\). \(f_n(1/2) = 0\) for all \(n\) and at every other point \(\lim\ f_n(x) = 1\)