2. Limits and convergence.

**ex2.4.7.** Let $f$ be a continuous function defined on a compact metric space. Use the characterization given in the previous exercise to prove that the range of $f$ is compact.

**ex2.4.8.** The range of a continuous map from a compact metric space into a metric space is compact.

**ex2.4.9.** A uniformly continuous function $f$ on $[0, 1]$ maps Cauchy sequences to Cauchy sequences.

**ex2.4.10.** Let $f = (f_1, \ldots, f_d)$ be an $\mathbb{R}^d$-valued function on $(X, \rho)$.

a. Prove that $f$ is continuous at a point $x_0 \in X$ if and only if every “component function” $f_j$, $j = 1, \ldots, d$, is continuous at $x_0$.

b. Prove that $f$ is uniformly continuous on $X$ if and only if every “component function” $f_j$, $j = 1, \ldots, d$, is uniformly continuous.

**2.4.6 Equicontinuity.** Let $\mathcal{F}$ be a set of continuous real-valued functions defined on a metric space $(X, \rho)$.

**Definition:** $\mathcal{F}$ is **equicontinuous at a point** $x \in X$ if for every $\epsilon > 0$ there exist $\delta = \delta(x, \epsilon)$ such that if $y \in B(x, \delta)$ then for every $f \in \mathcal{F}$, $|f(y) - f(x)| < \epsilon$.

$\mathcal{F}$ is **equicontinuous on** $X$ if it is equicontinuous at every $x \in X$.

2.5 Connectedness

**2.5.1 Definition:** A metric space $(X, \rho)$ is **connected** if it is not the union of two disjoint nonempty closed subsets.

Since the complement of an open set is closed and the complement of a closed set is open, we can define connectedness by:

$(X, \rho)$ is **connected** if it is not the union of two disjoint nonempty open subsets.

**Proposition.** An interval $I \subset \mathbb{R}$ is connected.

**Proof:** Proof by contradiction. We consider first the case that $I = [a, b]$ is bounded and closed. Assume $I = E_1 \cup E_2$, with $E_1$ and $E_2$ nonempty and closed, numbered so that $a \in E_1$.

Write $c = \inf E_2$. As $[a, c) \subset E_1$, $c$ is a limit point of $E_1$ and, as $E_1$ is closed, we have $c \in E_1$.

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On the other hand, being the infimum of the closed set $E_2$ we have $c \in E_2$, contradicting the assumption that $E_1$ and $E_2$ are disjoint.

For a general interval $J$, that may be open, half open, unbounded, we assume $J = E'_1 \cup E'_2$ with $E'_j$ nonempty, closed and disjoint, take $a \in E'_1$, $b \in E'_2$, write $I = [a, b], E_j = E'_j \cap I$, and apply the case where $I = [a, b]$ is bounded and closed.

**Theorem.** A continuous real-valued function defined on a connected space $X$ (e.g. on $[0, 1]$) has the intermediate value property, that is, if $x_0, x_0 \in X$, $f(x_0) = a \neq f(x_1) = b$ and $a < c < b$, then there is $y \in X$ such that $f(y) = c$.

**ex2.5.1.** A closed subset $E \subset \mathbb{R}$ is connected if and only if it is an interval. The interval can be bounded or extend to $+\infty$, or to $-\infty$ or to both.

**ex2.5.2.** An interval is characterized (among the subsets of $\mathbb{R}$) by the property: if $a$ and $b$ are points in it and $a < c < b$ then $c$ is in it as well.

**ex2.5.3.** The range of a real-valued continuous function on a connected metric space is an interval.

**ex2.5.4.** The range of a continuous map from a connected metric space into a metric space is connected.

## 2.6 Series of functions

### 2.6.1 Pointwise convergence.

We now consider sequences and series of (real-valued) functions defined on a finite interval, say on $[0, 1]$, or, more generally, on a metric space $(X, \rho)$.

The convergence of such a sequence or a series at a point $x_0$ is the convergence of a sequence or of a series of real numbers, which was studied earlier. Here we consider the **Pointwise convergence** of a sequence (or a series) of continuous functions, and the **uniform convergence** of such. Since the convergence of a series is defined by the convergence of the sequence of its partial sums (or, in the case of summability, of averages thereof) we focus on sequences.

**Definition:** The sequence $\{f_n\}$ converges **pointwise** on $[0, 1]$ if it converges at every point in the interval.
2. Limits and convergence.

We know that for real-valued functions convergence at a point \( x_0 \)
is equivalent to the sequence \( \{f_n(x_0)\} \) being a Cauchy sequence, i.e.,
\[
\forall \varepsilon > 0, \exists N = N(\varepsilon, x_0) \text{ such that if } n, m > N \text{ then } |f_n(x_0) - f_m(x_0)| < \varepsilon.
\]
The sequence converges pointwise if the condition is satisfied for every \( x_0 \in [0, 1] \).

2.6.2 Uniform convergence. In the context and notation of the previous subsection, assume that \( \{f_n(x)\} \) converges to \( f(x) \) everywhere
on \([0, 1] \).

**Definition:** The sequence \( \{f_n\} \) converges to \( f \) uniformly means:
\[
\forall \varepsilon > 0, \exists N = N(\varepsilon) \text{ such that if } x \in [0, 1] \text{ and } n > N.
\]
The uniformity is the fact that, for all \( \varepsilon > 0 \), there is a common \( N(\varepsilon) \)
which accommodates all the points \( x \in [0, 1] \).

**Example:** Consider an infinite series \( \sum_{n=0}^{\infty} g_n(x) \), the functions \( g_n \)
defined on \([0, 1] \) and satisfy \( |g_n(x)| \leq c_n \) with constants \( c_n \) such that \( \sum c_n \)
converges. Then the series \( \sum g_n \) converges uniformly (and absolutely).

**Theorem.** a. The sequence \( \{f_n\} \) converges uniformly on \([0, 1] \) if, and only if it is uniformly Cauchy, i.e., \( \forall \varepsilon > 0, \exists N = N(\varepsilon) \text{ such that for all } x \in [0, 1], \text{ if } n, m > N \text{ then } |f_n(x) - f_m(x)| < \varepsilon. \)

b. A uniform limit of a sequence \( \{f_n\} \) of continuous functions is continuous.

**Proof:** a. Assume that the sequence \( \{f_n\} \) converges uniformly on \([0, 1] \), and denote the limit function by \( f \). Then \( \forall \varepsilon > 0, \exists N = N(\varepsilon) \)
such that if \( n > N \) then, for all \( x \in [0, 1] \), we have \( |f_n(x) - f(x)| < \varepsilon/2. \)
If \( m > N \) then \( |f_m(x) - f(x)| < \varepsilon/2 \) for all \( x \in [0, 1] \), so that if both \( n, m > N, |f_n(x) - f_m(x)| < \varepsilon \), and \( \{f_n\} \) is uniformly Cauchy.

Conversely, assume that \( \{f_n\} \) is uniformly Cauchy. We know that the sequence converges pointwise. Denote \( f(x) = \lim_{n \to \infty} f_n(x) \); we claim that the convergence is uniform.

The sequence \( \{f_n\} \) being uniformly Cauchy means the following:
\( \forall \varepsilon > 0, \exists N = N(\varepsilon) \) such that \( |f_n(x) - f_m(x)| < \varepsilon \) for all \( x \) provided \( m, n > N \). Fixing \( m > N \) and taking the limit as \( n \to \infty \), the estimate becomes \( |f(x) - f_m(x)| < \varepsilon \), i.e., \( f_n \to f \) uniformly.

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b. Assume \( f(x) = \lim_{n \to \infty} f_n(x) \), the convergence uniform. The continuity of \( f \) is obtained by showing that \( o_f(x_0) = 0 \) for every \( x_0 \), making use of the following observation he proof of which is left as an exercise:

If \( g \) is continuous on \([0, 1]\) and \( |g(x) - f(x)| < \varepsilon \) for all \( x \in E \). Then \( o_f(x) < 2\varepsilon \) for all \( x \in [0, 1] \).

For every \( \varepsilon > 0 \) there exists \( n \) such that \( |f_n(x) - f(x)| < \varepsilon \) for all \( x \), which implies \( (f_n \) plays the role of \( g \) above) \( o_f(x) < 2\varepsilon \) for all \( x \in [0, 1] \).
Since \( \varepsilon > 0 \) is arbitrary we obtain \( o_f(x) = 0 \), and \( f \) is continuous at every \( x \in [0, 1] \).

---

2.6.1. Assume \( g \) continuous on \( E \) and \( |g(x) - f(x)| < \varepsilon \) for all \( x \in E \). Prove: \( o_f(x) < 2\varepsilon \) for all \( x \in E \).

2.6.2. Let \( \varphi_n \) be real-valued functions defined on a set \( E \). Assume that \( |\varphi_n(x)| \leq b_n \) for all \( x \in E \), and assume that \( \sum b_n < \infty \). Prove that \( \sum \varphi_n(x) \) converges uniformly on \( E \).

2.6.3 Equicontinuity A set \( \{f_n\} \) of real-valued functions on \([0, 1]\) is equicontinuous at a point \( x_0 \) if for every \( \varepsilon > 0 \) the exists \( \delta(\varepsilon) > 0 \) such that if \( |x_0 - y| < \delta \) then \( |f_n(x_0) - f_n(y)| < \varepsilon \) for all \( n \).

\( \{f_n\} \) is uniformly equicontinuous if for every \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) \) such that if \( |x - y| < \delta \) then \( |f_n(x) - f_n(y)| < \varepsilon \) for all \( n \) and for all \( x \).

The uniform continuity of a function is the fact that \( \delta(\varepsilon) \) can be chosen so as to accommodate all \( x \) for the given function. The equicontinuity is the fact that \( \delta(\varepsilon) \) can be chosen to accommodate all the functions \( f_n \) (either at a specific point or at every point).

### 2.7 Power Series

Power series are series of the form

\[
\sum_{n=0}^{\infty} a_n x^n.
\]

Recall that by definition \( x^0 = 1 \). The (real) numbers \( a_n \) are the coefficients of the series. The series certainly converge for \( x = 0 \); it may or may not converge elsewhere, and we shall be interested only in the case it does.

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2. Limits and convergence.

2.7.1 Convergence.

The domain of convergence of a power series \( \sum_{0}^{\infty} a_n x^n \) is given by the following parameter, called the radius of convergence:

\[
R = (\limsup |a_n|^{1/n})^{-1}.
\]

Theorem. If \( R = 0 \) then the series converges at the only point \( x = 0 \).
If \( R > 0 \) then the series converges for all \( x \in (-R, R) \) and uniformly so on every interval \( [-r, r] \) where \( r < R \). The series may or may not converge at the points \( x = R \) and \( x = -R \).

Proof: By its definition, \( R = \limsup \{ c : |a_n c^n| = O(1) \} \). If \( |x| > R \) then the terms \( a_n x^n \) are not bounded and the series does not converge. In particular, if \( R = 0 \) the same holds for any \( x \neq 0 \).

If \( R > 0 \) and \( r < c < R \) then \( |a_n c^n| \) is bounded, say \( |a_n c^n| < B \) for all \( n \in \mathbb{N} \). Now, \( |a_n r^n| \leq B (\frac{r}{c})^n \), and the uniform convergence for \( x \in [-r, r] \) is given by comparison to the geometric series.

2.7.2 Products of power series.

Proposition. Let \( \sum_{0}^{\infty} a_n = A \) and \( \sum_{0}^{\infty} b_n = B \) be absolutely convergent. Define \( c_n = \sum_{j+k=n} a_j b_k \). The series \( \sum_{0}^{\infty} c_n \) is absolutely convergent to sum \( AB \).

See subsection 2.3.6.

Let \( \sum a_n x^n \) and \( \sum b_n x^n \) be power series with positive radii of convergence \( R_a \) and \( R_b \) respectively. Let \( R = \min(R_a, R_b) \) and denote \( f(x) = \sum a_n x^n \) and \( g(x) = \sum b_n x^n \) in the respective domains of convergence, and in particular for \( |x| < R \). Denote \( c_n = \sum_{k+l=n} a_k b_l \).

Theorem. The radius of convergence of the series \( \sum c_n x^n \) is \( \geq R \), and, for \( x \in \{ x : |x| < R \} \),

\[
(2.7.2) \quad F(x) = \sum c_n x^n = f(x)g(x).
\]

Proof: This is an immediate application of the proposition.

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2.7.3 **Abel summability.** Given an infinite series of real numbers \( \sum_{n=0}^{\infty} a_n \), we consider the power series \( \sum_{n=0}^{\infty} a_n x^n \) and, assuming that the radius of convergence is at least 1, would like to understand the relation between the existence and the value of the limit \( \lim_{x \to 1^{-}} f(x) \) on the one hand and the convergence and the sum of \( \sum a_n \) on the other.

A theorem of Abel, below, states that if the series converges, and its sum is \( S \) then \( \lim_{x \to 1^{-}} f(x) = S \). On the other hand there exist series that do not converge but for which the limit \( \lim_{x \to 1^{-}} f(x) \) exists. We say that the series is Abel-summable to the value \( \lim_{x \to 1^{-}} f(x) \).

**Theorem (Abel).** Assume the series \( \sum_{n=0}^{\infty} a_n \) convergent. Then the function \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) converges, as \( x \to 1 \), to \( S = \sum_{n=0}^{\infty} a_n \).

**Proof:** Denote \( s_n = \sum_{n=0}^{n} a_j \) and \( s_n(x) = \sum_{n=0}^{n} a_j x^j \). The assumption is that \( s_n \to S \) as \( n \to \infty \). Now, if \( 0 < x < 1 \)

\[
(2.7.3) \quad f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{N} (s_j - s_{j-1}) x^j = \sum_{j=0}^{N} (s_j x^j - s_{j+1} x^j).
\]

Observe that \( x^j - x^{j+1} = x^j (1-x) > 0 \) and \( \sum x^j (1-x) = 1 \) so that \( f(x) \) is a **weighted average** of \( s_j \) with weights \( x^j (1-x) \). Now

\[
(2.7.4) \quad |f(x) - S| \leq \sum_{n=0}^{N} x^j (1-x) |s_j - S|.
\]

Denote \( S = 1 + \sup |s_n| \). Given \( \varepsilon > 0 \), let \( N = N(\varepsilon) \) be such that \( |s_n - S| < \varepsilon / 2 \) for \( n \geq N \).

With \( \varepsilon \) and \( N \) fixed there exists \( x_0 < 1 \) such that for \( x > x_0 \) we have

\[
\sum_{n=0}^{N} x^j (1-x) < \varepsilon / 4 S.
\]

Split the sum in (2.7.4) to the sums \( \sum_{n=0}^{N} \) and \( \sum_{n=N+1}^{\infty} \).

For the first sum, when \( x > x_0 \), we have \( \sum_{n=0}^{N} x^j (1-x) |s_j - S| < \varepsilon / 2 \).

For the second

\[
(2.7.5) \quad \sum_{n=N+1}^{\infty} x^j (1-x) |s_j - S| \leq \sup_{j > N} |s_j - S| \leq \varepsilon / 2.
\]

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