1.1. Prove $1^2 + 2^2 + \ldots + n^2 = \frac{1}{6} n (n+1)(2n+1)$ for all natural numbers $n$.

1.3. Prove $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for all natural numbers $n$.

1.4. (a) Guess a formula for $1 + 3 + \ldots + (2n - 1)$ by evaluating the sum for $n = 1$, $2$, $3$, and $4$. [For $n = 1$, the sum is simply $1$.]

(b) Prove your formula using mathematical induction.

1.6. Prove that $11^n - 4^n$ is divisible by $7$ when $n$ is a natural number.

1.8. The principle of mathematical induction can be extended as follows. A list $P_m$, $P_{m+1}$, ... of propositions is true provided (i) $P_m$ is true, (ii) $P_{n+1}$ is true whenever $P_n$ is true and $n \geq m$.

(a) Prove that $n^2 > n + 1$ for all integers $n \geq 2$.

(b) Prove that $n! > n^2$ for all integers $n \geq 4$. [Recall that $n! = n(n-1) \cdots 2 \cdot 1$; for example $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.]

1.11. For each $n \in \mathbb{N}$, let $P_n$ denote the assertion “$n^2 + 5n + 1$ is an even integer.”

(a) Prove that $P_{n+1}$ is true whenever $P_n$ is true.

(b) For which $n$ is $P_n$ actually true? What is the moral of this exercise?

1.12. For $n \in \mathbb{N}$, let $n!$ [read “$n$ factorial”] denote the product $1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$. Also let $0! = 1$ and define

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} \text{ for } k = 0, 1, \ldots, n.$$ 

The binomial theorem asserts that

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n$$

$$= a^n + n a^{n-1} b + \frac{1}{2} n(n-1) a^{n-2} b^2 + \ldots + n a b^{n-1} + b^n$$

(a) Verify the binomial theorem for $n = 1, 2, \text{ and } 3$.

(b) Show that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $k = 1, 2, \ldots, n$.

(c) Prove the binomial theorem using mathematical induction and part (b).

2.3. Show that $(2 + \sqrt{2})^{1/2}$ does not represent a rational number.

2.5. Show that $(3 + \sqrt{2})^{2/3}$ does not represent a rational number.