HW 8 Solutions
Math 115, Winter 2009, Prof. Yitzhak Katznelson

23.5 b) Prove that if \( \limsup |a_n| > 0 \), then the radius of convergence \( R \) of \( \sum a_n x^n \) is \( \leq 1 \).

Notice: by Theorem 23.1, it suffices to prove \( \limsup |a_n|^{1/n} \geq 1 \), since \( R \) is the reciprocal of that quantity.

To do this, notice the following. \( \limsup |a_n| > 0 \), so there exists an \( s \) with \( 0 < s < \limsup |a_n| \). Since \( \limsup |a_n| \) is the supremum of the set of subsequential limits, there is a subsequence \( a_{n_i} \) with \( |a_{n_i}| \) converging to \( t > s \). Thus there exists \( N \) for which \( n \geq N \) implies \( |a_{n_i}| > s \), and hence \( |a_{n_i}|^{1/n} > s^{1/n} \).

Take the lim sup of both sides (the strict inequality then turns into a \( \geq \) sign). Since \( s > 0 \), we know that \( \limsup s^{1/n} = \lim s^{1/n} = 1 \). So \( \limsup |a_{n_i}|^{1/n} \geq \limsup s^{1/n} = 1 \). This means that the sequence \( |a_{n_i}|^{1/n} \) has a subsequential limit \( t \geq 1 \) (\( t = \infty \) is possible). Since any subsequence of \( a_{n_i} \) is also a subsequence of \( a_n \), \( t \) is a subsequential limit of \( |a_n|^{1/n} \) as well; this means that \( \limsup |a_n|^{1/n} \geq t \geq 1 \), as desired. ■

23.7 a) If \( x \) is not a multiple of \( \pi \), then \( |\cos x| < 1 \), so \( \lim |\cos x|^n = 0 \) and hence \( \lim f_n(x) = \lim (\cos x)^n = 0 \). (We use the fact that if \( \lim |a_n| = 0 \), then \( \lim a_n = 0 \) - you should know how to prove this).

b) If \( x \) is an even multiple of \( \pi \), then \( \cos x = 1 \), so \( (\cos x)^n = 1 \) for all \( n \), so \( \lim f_n(x) = \lim (\cos x)^n = \lim 1 = 1 \).

c) If \( x \) is an odd multiple of \( \pi \), then \( \cos x = -1 \), so \( (\cos x)^n = (-1)^n \); then \( \lim f_n(x) = \lim (-1)^n \) does not exist.

24.12. Take a sequence of functions \( f_n \) on a set \( S \subseteq \mathbb{R} \) and a function \( f \) on \( S \). We want to prove that \( f_n \to f \) uniformly on \( S \) if and only if \( \lim_{n \to \infty} \sup \{ |f(x) - f_n(x)| : x \in S \} = 0 \).

First suppose \( f_n \to f \) uniformly on \( S \). Then pick \( \epsilon > 0 \). By the definition of uniform convergence, there is an \( N \) such that \( n \geq N \) implies \( |f(x) - f_n(x)| < \epsilon/2 \) for all \( x \in S \). Thus \( \epsilon/2 \) is an upper bound for \( |f(x) - f_n(x)| \), so for \( n \geq N \), we have \( \sup \{ |f(x) - f_n(x)| : x \in S \} \leq \epsilon/2 < \epsilon \). That supremum is non-negative, so its absolute value is also \( < \epsilon \). Thus what we’ve verified is precisely the definition of the limit, and so we’ve shown that \( \lim_{n \to \infty} \sup \{ |f(x) - f_n(x)| : x \in S \} = 0 \). (Notice: when taking the supremum, we don’t necessarily keep strict inequality, which is why we had to use \( \epsilon/2 \).

Conversely, suppose that \( \lim_{n \to \infty} \sup \{ |f(x) - f_n(x)| : x \in S \} = 0 \). Then \( \epsilon > 0 \). By the definition of the limit, there is an \( N \) such that \( n \geq N \) implies \( \sup \{ |f(x) - f_n(x)| : x \in S \} < \epsilon \). So for all \( n \geq N \)
and all \( x \in S \), \( |f(x) - f_n(x)| < \sup \{|f(x) - f_n(x)| : x \in S\} < \epsilon \). This is precisely the \( N \) we need for uniform convergence, and so \( f_n \to f \) uniformly. 

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24.13. Suppose \( f_n \) is a sequence of uniformly continuous functions on \((a, b)\) and \( f_n \to f \) uniformly on \((a, b)\); we want to show that \( f \) is uniformly continuous on \((a, b)\).

So: pick \( \epsilon > 0 \). Then by uniform convergence, there exists an \( N \) so that if \( n \geq N \) and \( x \in (a, b) \), \( |f_n(x) - f(x)| < \epsilon/3 \). In particular \( |f_N(x) - f(x)| < \epsilon/3 \) for all \( x \in (a, b) \).

Now \( f_N(x) \) is uniformly continuous. So there exists \( \delta > 0 \) so that if \( x, y \in (a, b) \) with \( |x - y| < \delta \), then \( |f_N(x) - f_N(y)| < \epsilon/3 \).

If \( x, y \in (a, b) \) with \( |x - y| < \delta \), then by the triangle inequality:

\[
|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|
\]

\[
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

Thus \( f \) is uniformly continuous. 

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24.14. Let \( f_n(x) = \frac{nx}{1 + nx^2} \).

a) We want to show that \( f_n \to 0 \) pointwise on \( \mathbb{R} \). First check it at 0: \( f_n(0) = 0 \), so \( f_n(0) \to 0 \). Now check it at some \( x \neq 0 \); we want to show that \( \frac{nx}{1 + nx^2} \to 0 \). Well, \( f_n(x) = \frac{x}{1/n + nx^2} \). The limit of the numerator is \( x \), and the limit of the denominator is \( \lim 1/n + \lim nx^2 \). Since \( x \neq 0 \), the second limit is \( +\infty \) while the first limit is 0, and so the limit of the denominator is \( +\infty \). This means that \( \lim f_n(x) = 0 \). So \( f_n \to 0 \) pointwise on \( \mathbb{R} \).

b) Does \( f_n \to 0 \) uniformly on \([0, 1]\)? Justify.

The answer is no. We claim that for all \( n \), \( \sup \{|f_n(x) - 0| : x \in [0, 1]\} = 1/2 \): if this is true, \( \lim \sup \{|f_n(x) - 0| : x \in [0, 1]\} = 1/2 \neq 0 \), so by 24.12 \( f_n \) does not converge to 0 uniformly.

To prove the claim, we’ll use one-variable calculus to find the maximum of \( f_n(x) \) on \([0, 1]\) and then show that it’s 1/2. So take the derivative:

\[
f_n'(x) = \frac{(1 + n^2x^2)(n) - (nx)(2nx)}{(1 + n^2x^2)^2} = \frac{n - n^3x^2}{(1 + n^2x^2)^2}.
\]

Setting this equal to 0 gives us the equation \( n - n^3x^2 = 0 \), or dividing by \( n \), \( 1 = n^2x^2 \); so there is a critical number at \( x = 1/n \). Also \( f_n'(x) \) is never infinity, since the denominator is always positive. To find the maximum and minimum, then, we just need to check the endpoints
and critical number: 0, 1, and 1/n. We find that:

\[ f_n(0) = 0, \quad f_n(1) = \frac{n}{1+n^2} \leq \frac{1}{2}, \quad f_n(1/n) = \frac{1}{2}, \]

and so \( \sup \{|f_n(x) - 0| : x \in [0, 1]\} = 1/2, \) so our claim is proven. (The inequality in the middle above is easy, as it's obviously true for \( n = 1, \) and for \( n \geq 2 \) we have \( f_n(1) < \frac{n}{n^2} = \frac{1}{n} \leq \frac{1}{2}. \) ■

26.1. Prove Theorem 26.4 for \( x > 0: \) that is for \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) with radius of convergence \( R > 0, \) we have for \( 0 < x < R \) that

\[ \int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}. \]

Proof: Fix \( x \) with \( 0 < x < R. \) On the interval \([0, x],\) the sequence of partial sums \( \sum_{k=0}^{n} a_k t^k \) converges uniformly to \( f(t) \) by Theorem 26.1. Consequently, by Theorem 25.2 (which lets us interchange integrals and uniform limits), we have that:

\[ \int_0^x f(t)dt = \lim_{n \to \infty} \int_0^x (\sum_{k=0}^{n} a_k t^k)dt = \lim_{n \to \infty} \sum_{k=0}^{n} a_k \int_0^x t^k dt \]

\[ = \lim_{n \to \infty} \sum_{k=0}^{n} a_k \frac{x^{k+1}}{k+1} = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}, \]

which is exactly what we wanted to show. ■

26.4. a) We know that \( e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \) for all \( x \in \mathbb{R}. \) Of course, for \( x \in \mathbb{R}, \) we have \( -x^2 \in \mathbb{R}, \) so the formula is valid when we replace \( x \) by \( -x^2. \) So: \( e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}. \)

b) Express \( F(x) = \int_0^x e^{-t^2} dt \) as a power series. To do this, replace \( e^{-t^2} \) by its power series summation: \( F(x) = \int_0^x \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} \right) dt. \) The thing inside the integral is still a power series (don’t be thrown off by the \( 2n \) in the exponent - this is just a power series where all the odd power terms are 0), so we may integrate it term-by-term, and see that:

\[ F(x) = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n}{n!} t^{2n} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1} = \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1}. \]

(Yes, the result is still a power series - skipping exponents doesn’t matter (adding fractional exponents would be a problem, but we don’t do that here). If you want to write it as \( \sum_{m=0}^{\infty} a_m x^m, \) then you can
easily check that the $a_m$ are given by the following: if $m$ is even, $a_m = 0$,
and if $m$ is odd, then $a_m = \frac{(-1)^{m-1}}{m(2m-1)!})$.

29.7. a) Suppose that $f$ is twice differentiable on an open interval $I$ with $f''(x) = 0$ for all $x \in I$. We want to show that $f$ has the form $f(x) = ax + b$ for suitable constants $a$ and $b$.

First, notice that $f'(x)$ has the property that $(f'(x))' = 0$ for all $x \in I$; by Corollary 29.4, $f'(x)$ must be a constant function; say $f'(x) = a$.

Then let $g(x) = ax$ on $I$. $f'(x) = a = g'(x)$ on $I$, so by Corollary 29.5, there is a constant $b$ such that $f(x) = g(x) + b = ax + b$ on $I$. That’s it. ■

b) Suppose that $f$ is three times differentiable on an open interval $I$ with $f'''(x) = 0$ on $I$. We claim that $f(x) = cx^2 + dx + e$ for some constants $c$, $d$, and $e$.

Indeed, notice that $f'(x)$ satisfies the hypothesis of a), so for some constants $a$ and $b$, $f'(x) = ax + b$. Let $g(x) = \frac{a}{2}x^2 + bx$; then $f'(x) = ax + b = g'(x)$ on $I$, so by Corollary 29.5, we have that for some constant $e$, $f(x) = g(x) + e = \frac{a}{2}x^2 + bx + e$. Letting $c = a/2$ and $d = b$ proves the claim. ■

34.2. a) Calculate $\lim_{x \to 0} \frac{1}{x} \int_0^x e^{t^2} dt$.

To do this, let $F(x) = \int_0^x e^{t^2} dt$, then we want to calculate $\lim_{x \to 0} \frac{F(x)}{x}$. Since $F(0) = 0$ (an integral from 0 to 0 of anything is 0), the thing we want to calculate can be re-written as $\lim_{x \to 0} \frac{F(x) - F(0)}{x}$. By definition, this is $F'(0)$. By the Fundamental Theorem of Calculus II, this is equal to $e^{0^2} = 1$. ■

b) Calculate $\lim_{h \to 0} \frac{1}{h} \int_3^{3+h} e^{t^2} dt$.

To do this, let $F(x) = \int_0^x e^{t^2} dt$ as above. Notice that our integral is actually $F(3 + h) - F(3)$. So we want to calculate $\lim_{h \to 0} \frac{F(3+h) - F(3)}{h}$, which by definition is $F'(3)$. Again by the Fundamental Theorem of Calculus II, this is $e^{3^2} = e^9$. ■

(A note about this problem: the integral $F(x)$ is literally impossible to evaluate directly! It’s not just that it’s hard, there is actually no formula for it. It can only be done by numerical approximation. Of course, as we’ve seen here, we can understand $F'(x)$ by using FTC II.)