1. Prove: If $A$ is a diagonal matrix with distinct entries on the diagonal, and if $B$ is a matrix such that $AB = BA$, then $B$ is diagonal.

2. Let $T \in \mathcal{L}(V)$, and $V = U \oplus W$ with both summands $T$-invariant. Let $\pi$ be the projection onto $U$ along $W$.
   a. Prove that $\pi$ commutes with $T$.
   b. Is $\pi$ necessarily of the form $P(T)$ for some polynomial $P$?
   c. Does $\pi$ commute with every operator that commutes with $T$?

3. If $\min P_T$ is irreducible, then $\min P_{T,v} = \min P_T$ for every $v \neq 0$ in $V$.

4. If $\min P_T$ is irreducible then $\dim V$ is divisible by $\deg \min P_T$.
   *Hint:* Use Proposition 5.3.2

5. Let $P_1, P_2 \in \mathbb{F}[x]$. Prove:
   \[ \ker(P_1(T)) \cap \ker(P_2(T)) = \ker(\gcd(P_1, P_2)). \]

6. *(Schur’s lemma).* A system $\{W, S\}$, $S \subset \mathcal{L}(W)$, is minimal if no nontrivial subspace of $W$ is invariant under every $S \in S$.
   Assume that $\{W, S\}$ is minimal, and $T \in \mathcal{L}(W)$.
   a. If $T$ commute with every $S \in S$, so does $P(T)$ for every polynomial $P$. 

b. If $T$ commutes with every $S \in S$, then $\ker(T)$ is either \{0\} or $W$. That means that $T$ is either invertible or identically zero.

c. With $T$ as above, the minimal polynomial $\min P_T$ is irreducible.

d. If $T$ commute with every $S \in S$, and the underlying field is $\mathbb{C}$, then $T = \lambda I$. 
*Hint:* The minimal polynomial of $T$ must be irreducible, hence linear.

7 Assume that $T$ is invertible and $\deg \min P_T = m$. Prove that

$$\min P_{T^{-1}}(x) = cx^m \min P_T(x^{-1}),$$

where $c = \min P_T(0)^{-1}$. 