1. Answer all the problems.

2. Write clearly! this refers both to the logic of your arguments and to your handwriting.

3. You may consult books, friends and relations until you feel ready to write the exam. Once you start writing, it is **closed book and no communications**.

4. The exam is due by noon, Friday June 5.

1. Consider the $n \times n$ matrix $A_n$ whose entries $a_{i,j}$ are equal to 1 if either $i = 1$ or $j = 1$ (or both), and $a_{i,j} = 0$ otherwise.
   a. Find the spectral norm $\|A_n\|_{sp}$, verify that it is a dominant eigenvector, and find the corresponding eigenvector.
   b. What are the characteristic and the minimal polynomials of $A_n$?
      *Hint:* Describe the range of $A_n$ (acting by multiplication on columns in $\mathbb{C}^n$).
   c. What of the properties of $A_n$ remain valid for any matrix with positive entries on the first column and on the first row, and zero coefficients everywhere else?

2. Given: $\mathcal{H}$ is a complex $n$-dimensional inner-product space, $S \in \mathcal{L}\mathcal{H}$. Prove that $S$ is normal if, and only if, the orthogonal complement of every $S$-invariant subspace is $S$-invariant.

3. Let $T \in \mathcal{L}\mathcal{V}$. Show that there exist vectors $v \in \mathcal{V}$ such that $\min P_{T,v} = \min P_T$.

4. Let $\mathcal{H}$ be a finite dimensional complex inner-product space, and $\mathbf{GL}(\mathcal{H})$ the multiplicative group of non-singular elements of $\mathcal{L}\mathcal{H}$.

   Prove that every finite subgroup $\mathcal{G} \subset \mathbf{GL}(\mathcal{H})$ is conjugate to a subgroup of $\mathcal{U}(\mathcal{H})$ (the group of unitary operators on $\mathcal{H}$). In other words: There exist some invertible $h \in \mathbf{GL}(\mathcal{H})$ such that $h^{-1}gh$ is unitary for every $g \in \mathcal{G}$.
   *Hint:* Show that $\mathcal{G}$ is a subgroup of the “unitary group” corresponding to some appropriate inner-product on $\mathcal{H}$, and explain how this proves the claim.
Nilpotent operators: An operator $T$ is nilpotent if $T^k = 0$ for some integer $k$, the smallest such $k$ is the order of $T$. The height of a vector $v$ (relative to $T$) is, by definition, the smallest integer $l$ such that $T^l v = 0$.

Assume $T \in \mathcal{L}V$ is nilpotent.

a. Show that if $W$ is a $T$-invariant subspace, then the operator $\overline{T}$ induced by $T$ on $V/W$ is nilpotent, and the order of $\overline{T}$ is not bigger than that of $T$.

b. Let $v_1 \in V$ be of height $k$, i.e., $T^k v_1 = 0$, and $T^{k-1} v_1 \neq 0$. Prove that the vectors $\{T^j v_1\}_{j=0}^{k-1}$ are linearly independent, and their span $V_1$ is $T$-invariant. What is the matrix of $T|V_1$ relative to the basis $u_j = T^{j-1} v_1$, $j = 1, \ldots, k$?

Denote by $S_n$ the group of permutations of $\{1, \ldots, n\}$. Let $\{e_j\}_{j=1}^n$ be a basis for $\mathbb{C}^n$.

a. Let $\sigma \in S_n$, and $A_\sigma$ the $n \times n$ operator which maps $e_i$ onto $e_{\sigma(i)}$. Describe the spectrum and the eigenvectors of $A_\sigma$ in terms of the cycle decomposition of $\sigma$.

b. Let $a_i > 0$, and let $B$ be the operator defined by $e_i \mapsto a_i e_{i+1}$ for $i < n$, $e_n \mapsto a_n e_1$. What is the spectrum of $B$?