This course, as you’ve discovered, is probably not like other math courses you’ve taken. Instead of doing problems and coming up with numerical answers or equations, you’re being asked to do proofs. The most important thing in a proof is whether it is logically correct. Your steps need to form a clear chain of reasoning leading from the hypotheses (if any) to the conclusion; if even one step is not true, that means the proof doesn’t work. If you skip steps and they are important, that can also be a problem, so please pay attention to detail and completeness; it’s most often better to err on the side of writing more, especially when time is not a factor. Whoever is grading your work needs to be able to understand your reasoning, so clarity is also important. Just try to be as clear, precise, complete, and correct as possible and you should be fine.

**Proof by Contradiction**

Probably the most obvious way to prove something is what’s called *direct proof*, where you start with the hypothesis and proceed through a series of steps directly to the conclusion. However, there are other methods of proving a statement; the most important one is called *proof by contradiction*. The basic idea of proof by contradiction is simple. You are asked to prove a statement of some sort, so just assume, instead, that it is not true and then derive a contradiction. Here is a famous example of proof by contradiction; we will use it to show that $\sqrt{2}$ is irrational without using the Rational Roots Theorem. Two simple facts first:

**Fact 1**: If $a$ is an integer, then $a$ is even if and only if $a^2$ is even.

**Fact 2**: Any rational number may be written in lowest terms, where the only common factors of the numerator and denominator are $\pm 1$.

**Theorem 1.** $\sqrt{2}$ is irrational.

*Proof.* Suppose for contradiction that $\sqrt{2}$ is a rational number. By Fact 2, it may be written in lowest terms as $\frac{a}{b}$, where $a$ and $b$ are integers. Consider the equation $\sqrt{2} = \frac{a}{b}$; this is the same as $b\sqrt{2} = a$. Square both sides to obtain $2b^2 = a^2$. But now $a^2$ is even. By Fact 1, $a$ is even, so there is an integer $c$ such that $a = 2c$. Plugging this into the equation, we get that $2b^2 = 4c^2$, or equivalently $b^2 = 2c^2$. But then $b^2$ is even. By Fact 1, $b$ is even.
But now we see that both $a$ and $b$ are even. However, $\frac{a}{b}$ was in lowest terms. This is a contradiction. Therefore $\sqrt{2}$ cannot be rational. 

So this is an example of proof by contradiction - all it is is assuming the statement you’re trying to prove is false and then showing that leads to a logical contradiction. Of course, you can’t get started on it until you realize exactly what the negation of the statement you’re trying to prove is! For example, when you are trying to prove, say, that $f(x) = g(x)$ for all $x$, the negation is that for SOME $x$, $f(x) \neq g(x)$. Remember these things and be careful with the “every” versus “some” distinction especially.

There are many examples of proof by contradiction in the solution sets to the first two homeworks; I have tried to clearly indicate it wherever it’s being used, so you should be able to find the examples easily.

**Opposing Inequalities**

In analysis, you’re often asked to prove that two quantities $A$ and $B$ are equal to each other. If you can’t come up with a quick direct proof, sometimes it is easiest to do the proof by “opposing inequalities.” That is, instead of proving directly that $A = B$, first prove that $A \leq B$ and then prove that $B \leq A$. This shows that $A = B$.

This trick can often be used in combination with contradiction - I find that sometimes contradiction can be effective in dealing with inequalities. It also shows up in disguise sometimes - in fact, the standard way of proving things involving sup (by showing it’s an upper bound, and then showing it’s the least upper bound) is very closely related to this method.

**Style**

A note on style. You will not be graded directly on proof style in this class, but it can be a very important tool in achieving the kind of clarity and correctness that you want. You may have seen the book trying to use as few English words as possible, and I’m not sure that’s always the best idea. This is a matter of personal preference, but I like to try to use enough English words to make the proof “grammatically correct” - that is, if you read it out loud, it should be a series of complete grammatical English sentences (of course this is not always achievable, but it’s a nice goal). Think of it as explaining your reasoning to a friend, and write down exactly what you’d explain. This style doesn’t work for everybody, and you shouldn’t be worried if you prefer something else, but it works for me and so I thought it could be a useful suggestion.
Symbols are useful, though, and you should think of them as shortcuts to avoid writing things out; for example, $\exists$ is just shorthand for “there exists.” As you continue to read the textbook and do problems, you’ll become more comfortable with the symbols and what they mean, and soon you’ll end up using more of these shorthand notations.

One thing that doesn’t matter at all is form. As long as you’re clear and understandable, it doesn’t matter whether you use a new line for each equation, or whether you write things in paragraph form, or whether you end your proof with “Q.E.D” or a blank square or a filled-in square. This is not high school geometry, so don’t worry about things like these!