Additional Problems for PS1

April 4, 2008

1. Show that the following are metric spaces:
   (a) \((X, d)\), where 
   \[ d(x, y) = \begin{cases} 
   0 & \text{if } x = y, \\
   1 & \text{if } x \neq y. 
   \end{cases} \]
   (b) \((\mathbb{R}, \rho_1)\), where 
   \[ \rho_1(x, y) = \begin{cases} 
   0 & \text{if } x = y, \\
   |x| + |y| & \text{if } x \neq y. 
   \end{cases} \]
   (c) Generalize the previous example: For \(x \neq y\), let \(\rho_n(x, y)\) be the sum of the Euclidean distances from \(x\) and \(y\) to the origin in \(\mathbb{R}^n\). If \(x = y\), define \(\rho_n(x, y)\) as zero. For example, \(\rho_2((1, 1), (2, 0)) = \sqrt{2} + 2\). Show that \((\mathbb{R}^n, \rho_n)\) is a metric space.

2. Let \(\mathbb{Q}^n\) be the set of ordered \(n\)-tuples of rational numbers. Use an inductive argument to show that \(\mathbb{Q}^n\) is countable.

3. Show that \(\sum_{i=1}^{n} i^2 = \frac{(n)(n+1)(2n+1)}{6}\).

4. Explain why the following inductive argument fails to show that all horses are the same color:
   Base case: There is at least one horse.
   Inductive step: Suppose that any set of \(n - 1\) horses is the same color, and consider a herd of \(n\) horses in a corral. Remove one horse from the corral, leaving \(n - 1\) horses there. By hypothesis, the remaining \(n - 1\) horses are all the same color. Put the removed horse back in the corral and remove a different horse. The remaining \(n - 1\) horses are again all the same color, so this proves that all horses are the same color.

5. **Bonus Problem** A colony of ommetaphobic logicians lives on a remote island. In general they lead a happy existence, but there is a strong taboo in the colony against knowing your own eye color. There are no mirrors on the island, and the polite logicians never discuss eye color. The rule in this community is that if you ever find out what color your eyes are, you proceed to the volcano in the center of the island and throw yourself in at dawn the next morning.

One day a sailor with a leaky boat appears on the shore of the island, and the logicians take care of him while he repairs his boat. After a week, the sailor is ready to depart. As he pulls away from the shore he thanks his hosts and remarks, “I was surprised to meet people with purple eyes, but it’s lucky for me that you were here. Goodbye!”
If there are $n$ purple-eyed logicians living on the island, what will happen?