HOMEWORK 2 SOLUTION

2.2. Solution: We need to show that, if \( a_1, a_2, \ldots, a_n \) are scalars such that \( a_1(v_1 - v_2) + a_2(v_2 - v_3) + \cdots + a_{n-1}(v_{n-1} - v_n) + a_nv_n = 0 \), then \( a_1 = a_2 = \cdots = a_n = 0 \).

Rearrange to get \( a_1v_1 + (a_2 - a_1)v_2 + \cdots + (a_n - a_{n-1})v_{n-1} + a_nv_n = 0 \). Since \( v_1, v_2, \ldots, v_n \) are linearly independent, we must have \( a_1 = 0, a_2 - a_1 = 0, a_3 - a_2 = 0, \ldots, a_n - a_{n-1} = 0 \). Hence \( a_1 = 0 \), and for all \( 2 \leq i \leq n \), \( a_i = a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_i - a_{i-1}) = 0 \). We have proved that \( a_1 = \cdots = a_n = 0 \). Therefore \( v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n \) are linearly independent.

2.3. Solution: Since \( (v_1 + w, v_2 + w, \ldots, v_n + w) \) is linearly dependent, there exists scalars \( a_1, a_2, \ldots, a_n \), not all zero, such that \( a_1(v_1 + w) + a_2(v_2 + w) + \cdots + a_n(v_n + w) = 0 \). This is equivalent to \( (a_1 + a_2 + \cdots + a_n)v_n + (a_1 + a_2 + \cdots + a_n)w = 0 \).

Since \( a_1, a_2, \ldots, a_n \) are not all zero, by the linear independence of \( (v_1, \ldots, v_n) \), it follows that \( a_1 + a_2 + \cdots + a_n \neq 0 \). Therefore \( (a_1 + a_2 + \cdots + a_n)w = -a_1v_1 - a_2v_2 - \cdots - a_nv_n \neq 0 \). Hence we must have \( a_1 + a_2 + \cdots + a_n \neq 0 \) (otherwise \( a_1 + a_2 + \cdots + a_n = 0 \) implies \( v_1, v_2, \ldots, v_n \) are linearly independent).

Dividing \( a_1 + a_2 + \cdots + a_n \) on both sides, we get \( w = -\frac{a_1}{a_1 + a_2 + \cdots + a_n}v_1 - \cdots - \frac{a_n}{a_1 + a_2 + \cdots + a_n}v_n \). Hence \( w \) is in the span of \( v_1, \ldots, v_n \).

2.7. Solution: We first prove that, if \( V \) is an infinite dimensional vector space, then there exists a sequence of vectors \( v_1, v_2, \ldots \), such that for any \( n, (v_1, \ldots, v_n) \) is linearly independent.

Being infinite dimensional, \( V \) is not the trivial vector space. Pick \( v_1 \) to be any nonzero vector in \( V \). Suppose we have already picked \( v_1, v_2, \ldots, v_m \), such that for any \( 1 \leq n \leq m, (v_1, \ldots, v_m) \) is linearly independent. We pick \( v_{m+1} \) as follows. Being infinite dimensional, \( (v_1, \ldots, v_m) \) does not span \( V \). Hence we can pick a vector \( v_{m+1} \) not in the span of \( v_1, \ldots, v_m \). This means that \( (v_1, \ldots, v_{m+1}) \) is also linear independent. Continue doing in this way, we get a sequence of vectors \( v_1, v_2, \ldots \), such that \( (v_1, \ldots, v_n) \) is linearly independent for any \( n \).

We then prove that, if there exists a sequence of vectors \( v_1, v_2, \ldots \) in \( V \), such that \( (v_1, \ldots, v_n) \) is linearly independent for every \( n \), then \( V \) is infinite dimensional. Suppose \( V \) is an \( m \)-dimensional vector space.

**Lemma 1.** Let \( V \) be an \( n \)-dimensional vector space. Then for any \( m > n \) and \( m \) vectors \( v_1, \ldots, v_m \) in \( V \), \( v_1, \ldots, v_m \) are linearly dependent.

**Proof.** Suppose \( v_1, \ldots, v_m \) are linearly independent, then by Theorem 2.12, \( v_1, \ldots, v_m \) can be extended to a basis of \( V \). But this would imply that \( \dim V \geq m \), a contradiction. \( \square \)

It follows from Lemma 1 that \( v_1, v_2, \ldots, v_{m+1} \) are linearly dependent, being \( m+1 \) vectors in an \( m \)-dimensional vector space. But this contradicts the hypothesis that \( (v_1, \ldots, v_n) \) is linearly independent for all \( n \).

To sum up, the proof is complete.

2.8. Solution: Let \( (x_1, x_2, x_3, x_4, x_5) \) be a vector in \( U \), then \( x_1 = 3x_2, x_3 = 7x_4 \).

Suppose \( t = x_2, s = x_4, u = x_5 \), then \( x_1 = 3t, x_3 = 7x_4 = 7t \), hence
Suppose \( (x_1, x_2, x_3, x_4, x_5) = (3t, 7s, s, u, u) = t(3, 1, 0, 0, 0) + s(0, 0, 7, 1, 0) + u(0, 0, 0, 0, 0) \).

Hence every vector in \( U \) is in the span of \( (3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 0) \). It is easy to check that \( (3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 0) \) is linearly independent.

We show that \( ((3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 0)) \) is linearly independent.

Suppose \( a, b, c \) are scalars such that \( a(3, 1, 0, 0, 0) + b(0, 0, 7, 1, 0) + c(0, 0, 0, 0, 1) = 0 \), then \( (3a, 7b, b, c) = 0 \). Hence \( a = b = c = 0 \), and it follows that \( ((3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 0)) \) is linearly independent.

To sum up, \( ((3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 0)) \) is a basis for \( U \).

2-9. Solution: We prove that there exists a basis \( (p_0, p_1, p_2, p_3) \) of \( \mathcal{P}_3(\mathbb{F}) \), such that none of \( p_0, p_1, p_2, p_3 \) is of degree 2, thereby answering the problem positively. In particular, we show that \((1, x, x^2 + x^3, x^3)\) is a basis for \( \mathcal{P}_3(\mathbb{F}) \).

First show that they span \( \mathcal{P}_3(\mathbb{F}) \). It suffices to note that \( ax^3 + bx^2 + cx + d = (a - b)x^2 + b(x^3 + x^2) + cx + d \).

We then show that they are linearly independent. Suppose \( a + bx + c(x^2 + x^3) + dx^3 = 0 \), then \( a + bx + cx^2 + (c + d)x^3 = 0 \), hence \( a = b = c = d = 0 \), i.e. \( a = b = c = d = 0 \). Thus these polynomials are linearly independent.

To sum up, the proof is complete.

2-11. Solution: Let \( n = \text{dim } U = \text{dim } V \), and let \( v_1, \ldots, v_n \) be a basis for \( U \). Then \( (v_1, \ldots, v_n) \) is linearly independent. Suppose \( U \subseteq V \), then there exists \( v_{n+1} \in V - U \). Since \( v_{n+1} \) is not in \( U \), it is not in the span of \( v_1, \ldots, v_n \). Therefore \( (v_1, \ldots, v_{n+1}) \) is linearly independent. By Theorem 2.12, \((v_1, \ldots, v_{n+1})\) can be extended to a basis of \( V \), thus \( \text{dim } V \geq n + 1 > n \), a contradiction. Hence we must have \( U = V \).

2-15. Solution: We give a counterexample. Let \( V = \mathbb{R}^3 \) be the 3-dimensional space. Let \( U_1 = \{ (x, y, z) \mid y = 0 \} \) be the \( xy \)-plane, \( U_2 = \{ (x, y, z) \mid z = 0 \} \) be the \( xz \)-plane and \( U_3 = \{ (x, y, z) \mid x = y = z \} \) be a line. Then it is easy to verify that in the equation of the problem, the dimensions from left to right are \( 3, 2, 2, 1, 1, 0, 0, 0, 0 \), but \( 3 \neq 2 + 2 + 1 - 1 - 0 - 0 + 0 \).

A1. Solution: For \( 1 \leq i < j \leq n \), let \( A_{ij} \) be the matrix such that its \((i, j)\) and \((j, i)\) entry are 1, -1, and all other entries are zero. There are \( n(n - 1)/2 \) such \( A_{ij} \)'s (choosing \( 1 \leq i < j \leq n \) is the same as choosing two numbers from \( \{1, 2, \ldots, n\} \), and we know that there are \( \binom{n}{2} \) \( n(n - 1)/2 \) ways to do so). We show that these \( A_{ij} \)'s form a basis for \( W \).

First show they span \( W \). Let \( B = (b_{ij})_{1 \leq i, j \leq n} \) be a skew-symmetric matrix. It follows from the skew-symmetric property that \( b_{ij} = -b_{ji} \), hence \( b_{ii} = 0 \) for all \( 1 \leq i \leq n \). It also follows from the skew-symmetric property that \( b_{ij} = -b_{ij} \), for all \( 1 \leq i < j \leq n \). Hence \( B = \sum_{1 \leq i, j \leq n} b_{ij}A_{ij} \). Hence the \( A_{ij} \)'s span \( W \).

We now show that the \( A_{ij} \)'s are linearly independent. Suppose \( \sum_{1 \leq i, j \leq n} a_{ij}A_{ij} = 0 \). Since the left hand side is a matrix whose \((i, j)\) entry is \( a_{ij} \), we must have that \( a_{ij} = 0 \) for all \( 1 \leq i < j \leq n \). This proves the linear independence.

To sum up, the dimension of \( W \) is \( n(n - 1)/2 \).

A2. Solution: Let \( (x_1, x_2, x_3, x_4, x_5) \) be a vector in \( W \), then \( x_1 - x_3 - x_4 = 0 \). Suppose \( t = x_2, s = x_3, u = x_4, w = x_5, \) then \( x_1 = x_3 + x_4 = s + u \), hence
(x_1, x_2, x_3, x_4, x_5) = (s + t, s, u, w) = s(1, 0, 1, 0, 0) + u(1, 0, 0, 1, 0) + t(0, 1, 0, 0, 0) + w(0, 0, 0, 0, 1). Hence every vector in W is in the span of (1, 0, 1, 0, 0), (1, 0, 0, 1, 0), (0, 1, 0, 0, 0), (0, 0, 0, 1, 0).

It is easy to check that (1, 0, 1, 0, 0), (1, 0, 0, 1, 0), (0, 1, 0, 0, 0), (0, 0, 0, 1, 0) ∈ W.

We show that (1, 0, 1, 0, 0), (1, 0, 0, 1, 0), (0, 1, 0, 0, 0), (0, 0, 0, 0, 1) is linearly independent. Suppose a, b, c, d ∈ C are scalars such that a(1, 0, 1, 0, 0) + b(1, 0, 0, 1, 0) + c(0, 1, 0, 0, 0) + d(0, 0, 0, 0, 1) = 0, then (a + b, c, a, b, d) = 0. Hence a = b = c = d = 0, and it follows that (1, 0, 1, 0, 0), (1, 0, 0, 1, 0), (0, 1, 0, 0, 0), (0, 0, 0, 0, 1) is linearly independent.

To sum up, (1, 0, 1, 0, 0), (1, 0, 0, 1, 0), (0, 1, 0, 0, 0), (0, 0, 0, 0, 1) is a basis for W.

A3. Solution: (a) U_2 is a finite dimensional vector space and U_1 ∩ U_2 is a subspace of U_2. It follows from Proposition 2.15 that dim(U_1 ∩ U_2) ≤ dim U_2 = n.

(b) It follows from Theorem 2.18 that dim(U_1 + U_2) = dim U_1 + dim U_2 − dim(U_1 ∩ U_2) ≤ dim U_1 + dim U_2 = m + n.

(c) It follows from Theorem 2.18 that dim(U_1 + U_2) = dim U_1 + dim U_2 − dim(U_1 ∩ U_2) = m + n − dim(U_1 ∩ U_2). Hence dim(U_1 + U_2) = m + n if and only if dim(U_1 ∩ U_2) = 0, which is equivalent to U_1 ∩ U_2 = {0}. But by Proposition 1.9, this happens if and only if U_1 + U_2 is a direct sum. Hence our conclusion is, dim(U_1 + U_2) = m + n if and only if U_1 ∩ U_2 = {0}, i.e. U_1 + U_2 is a direct sum.

References