Problem 1. Assume \( E \to M \) is a holo bd with a hermitian metric, and \( \nabla \) its Chern connection. Let \( \tau = (\tau_{ij}) \) denote the connection matrix in a local frame \( \{ e_k \} \) of \( E \).

(a) show that if \( \{ e_k \} \) is a holo frame, then

\[
\tau = \partial H \cdot H^{-1}
\]

where \( H = (h_{ij}) \) is the matrix of coefficients associated to \( h \) in this frame.

(b) show that if \( \{ e_k \} \) is a unitary frame (i.e. \( h_{ij} = \delta_{ij} \)), then \( \tau \) is skew hermitian:

\[
\tau_{ij} + \overline{\tau}_{ji} = 0 \quad \text{i.e.} \quad \tau + \tau^T = 0
\]

Conclude that the curvature matrix \( \Theta \) is skew-hermitian (i.e. takes values in the Lie algebra \( \text{u}(r) \)).

Problem 2. Assume \( E \) and \( F \) are cx v bd over \( M \) with connections \( \nabla^E \) and \( \nabla^F \).

(a) Show that

\[
\nabla(\sigma \oplus s) = \nabla^E(\sigma) + \nabla^F(s)
\]

\[
\nabla(\sigma \otimes s) = \nabla^E(\sigma) \otimes s + \sigma \otimes \nabla^F(s)
\]

\[
(\nabla T)(\sigma) = \nabla^F(T(\sigma)) - T(\nabla^E \sigma)
\]

for all \( \sigma \in \Gamma(E) \), \( s \in \Gamma(F) \) and \( T \in \Gamma(\text{Hom}(E, F)) \) define connections on \( E \oplus F \), \( E \otimes F \) and respectively \( \text{Hom}(E, F) \). What is the formula for the induced connection on \( E^* \)?

(b) show that any connection \( \nabla \) on the direct sum \( E \oplus F \) "projects" to a connection on \( E \):

\[
\nabla^E(s) = \pi_E \circ \nabla(s) \quad \text{for all} \quad \sigma \in \Gamma(E)
\]

and this construction is compatible with the one in (a) i.e. if we start with \( \nabla^E \) and \( \nabla^F \), induce \( \nabla \) on \( E \oplus F \) and then project back to the factors we get what we started with.

(c) if \( E, F \) are holo v. bd with a hermitian metric and Chern connections \( \nabla^E \) and \( \nabla^F \), show that the Chern connections of \( E \oplus F \), \( E \otimes F \), \( E^* \) and \( \text{Hom}(E, F) \) are given by (a).

Problem 3. Assume \( E \to M \) is a holo bd, and let \( A \) denote the space of connections \( \nabla \) on \( E \) compatible with the holo structure.

(a) Show that \( A \) is an affine space modeled on \( \Lambda^{1,0}(\text{End}(E)) \) i.e. the difference \( a = \nabla_1 - \nabla_2 \) between two such connections is a \((1, 0)\) form with values in \( \text{End}(E) \).

(b) Show that the difference between the dual connections on \( E^* \) is \( -a^* \).

(c) Calculate the corresponding difference of connections on \( E \oplus F \), \( E \otimes F \) and respectively \( \text{Hom}(E, F) \) in terms of the differences \( a_E \) and \( a_F \).

Problem 4. Assume \( E \to M \) is a holo v. bd. For any hermitian metric \( h \) on \( E \), consider its Chern connection \( \nabla \). Show that its curvature gives a class

\[
[R_{\nabla}] \in H^{1,1}(M, E)
\]

in Dolbeault cohom which is independent of the hermitian metric (called the Atiyah class).

Hint: use Bianchi identity \( \nabla R_{\nabla} = 0 \); relate the difference of curvatures to that of the connections.

Problem 5. Assume \( E, F \) are (cx) v. bds with connections. Given the curvatures \( R_{\nabla^E} \) and \( R_{\nabla^F} \), calculate the curvatures of induced connections on \( E \oplus F \), \( E \otimes F \), \( E^* \), \( \det E \).
Problem 6. Assume \( E \to M \) is a \( \mathbb{C} \) v. bd, \( F \subset E \) a \( \mathbb{C} \) subbundle and \( \pi : E \to E/F \) the quotient. Fix a connection \( \nabla \) and a hermitian metric \( h \) on \( E \), which induce ones on \( F \) and \( E/F \).

(a) denote by \( F^\perp \) the orthogonal complement of \( F \) in \( E \) wrt \( h \). Show that as hermitian v. bds

\[
F^\perp = E/F \quad \text{and} \quad E = F \oplus F^\perp.
\]

i.e. \( h \) defines a canonical splitting of the SES \( 0 \to F \to E \to E/F \to 0 \) as hermitian v. bds.

(b) show that

\[
A : \Gamma(F) \to \Lambda^1(E/F) \quad \text{defined by} \quad A(\sigma) = \pi(\nabla(\sigma)) \quad \text{for all} \quad \sigma \in \Gamma(F)
\]
is tensorial, i.e. can be regarded as a 1-form on \( M \) with values in \( \text{Hom}(F, E/F) \) (called the 2nd fundamental form of \( F \) in \( E \)).

(c) show that under the splitting \( E = F \oplus F^\perp \) (induced by \( h \)), the second fund form is

\[
A(\sigma) = \nabla^E(\sigma) - \nabla^F(\sigma) \quad \text{for all} \quad \sigma \in \Gamma(F),
\]

and \( A \) can be regarded as a 1-form on \( M \) with values in \( \text{Hom}(F, F^\perp) \).

(d) show that in a unitary frame on \( E \) (compatible with the splitting \( E = F \oplus F^\perp \)), the connection matrices \( \tau \) are related by

\[
\tau_E = \begin{pmatrix} \tau_F & \overline{A}^T \\ A & \tau_{F^\perp} \end{pmatrix}
\]

while for the curvature matrices \( \Theta_F = \Theta_E|_F + \overline{A}^T \wedge A \) and \( \Theta_{F^\perp} = \Theta_E|_{F^\perp} + A \wedge \overline{A}^T \).

Problem 7. Assume \( E \) is a holo v bd with a hermitian metric \( h \) and Chern connection \( \nabla \). Assume \( F \subset E \) a holo subbundle, and let \( F^\perp \) the orthogonal complement of \( F \) in \( E \) wrt \( h \). (Careful: in general, \( F^\perp \) may NOT be a holo sub-bd).

(a) show that the projection of \( \nabla \) to \( F \) is the Chern connection on \( F \) and that the second fundamental form \( A_F \) of \( F \) in \( E \) is of type \((1,0)\).

(b) show that \( F^\perp \) is a holo subbundle iff the second fundamental form of \( F^\perp \) has type \((1,0)\).