Problem 1
If $G$ has no odd cycles, then it is 2-colorable. Otherwise let $C$ be an odd cycle in $G$. Let $G'$ be the graph obtained by removing the vertices of $C$ from $G$. $G'$ has no odd cycles because any odd cycle in $G'$ would be an odd cycle in $G$ with no vertices in $C$. Therefore, we can 2-color $G'$. We know $C$ is three colorable, so we can extend our 2-coloring of $G'$ to a 5-coloring of $G$ by using three new colors to color $C$.

Problem 2
First we show that odd cycles are 3-critical. In the previous homework we showed that odd cycles have chromatic number three. Removing any vertex or edge from a cycle results in a tree (which is 2-colorable). Therefore odd cycles are 3-critical.

A graph with no odd cycles is 2-colorable, therefore any graph with chromatic number three contains an odd cycle. In general, however, we know that a $k$-critical graph cannot contain another $k$-critical graph as a proper subgraph. Therefore only odd cycles are 3-critical.

Problem 3
First we show that $\chi(G_1 \lor G_2) \leq \chi(G_1) + \chi(G_2)$ by exhibiting a coloring of $G_1 \lor G_2$ with $\chi(G_1) + \chi(G_2)$ colors. We color the vertices of $G_1$ using $\chi(G_1)$ colors and then color the vertices of $G_2$ using $\chi(G_2)$ colors each different from the ones we have used already. Then adding edges between $G_1$ and $G_2$ cannot join two vertices of the same color, so we have a valid coloring.

Next we show that $\chi(G_1 \lor G_2) \geq \chi(G_1) + \chi(G_2)$ by contradiction. Suppose there is a coloring of $G_1 \lor G_2$ with fewer colors. No color can be used in both $G_1$ and $G_2$ because of all the edges connecting these graphs. Therefore, either the vertices of $G_1$ are colored with fewer than $\chi(G_1)$ colors or the vertices of $G_2$ are colored with fewer than $\chi(G_2)$ colors. Either is a contradiction.

Therefore $\chi(G_1 \lor G_2) = \chi(G_1) + \chi(G_2)$.

Now suppose $G_1$ and $G_2$ are critical. Suppose edge $e$ is removed from $G_1 \lor G_2$. If $e$ is in $G_1$, then $(G_1 \lor G_2) - e = (G_1 - e) \lor G_2$, so by the above equality, $(G_1 \lor G_2) - e$ can be colored with fewer than $\chi(G_1 \lor G_2)$ colors. The same is true if $e$ is in $G_2$. Otherwise $e$ joins a vertex $v_1$ from $G_1$ and a vertex $v_2$ from $G_2$. We can color $(G_1 - v_1) \lor (G_2 - v_2)$ with no more than $\chi(G_1 \lor G_2) - 2$ colors by the above equality, so we can extend this to a coloring of $G_1 \lor G_2$ with no more than $\chi(G_1 \lor G_2) - 1$ colors by introducing one new color and using it for both $v_1$ and $v_2$. Therefore $G_1 \lor G_2$ is critical. (It suffices to consider removal of each edge because any proper subgraph of a connected graph is a subgraph of that graph minus some edge.)

Conversely, suppose $G_1 \lor G_2$ is critical. Because $G_1 \lor G_2 = G_2 \lor G_1$, if we show $G_1$ is critical then $G_2$ is also critical. Suppose then that $G_1$ is not critical.
Then there is a proper subgraph of $G'$ of $G_1$ that has chromatic number $\chi(G_1)$. But then $G'_1 \lor G_2$ is a proper subgraph of $G_1 \lor G_2$ that has chromatic number $\chi(G_1 \lor G_2)$. This contradicts that fact that $G_1 \lor G_2$ is critical. Therefore $G_1$ and $G_2$ are critical.

**Problem 4**

Notice that if $G$ is critical with edge $e$, then in any $\chi(G) - 1$ coloring of $G - e$ the endpoints of $e$ have the same color. Otherwise the $\chi(G) - 1$ coloring could extend to a valid coloring of $G$.

Now we will show that $(G_1 - vv_1) \lor (G_2 - vv_2) + v_1v_2$ is $k$-chromatic. If it were $k - 1$ colorable, then we could restrict any $k - 1$ coloring to a $k - 1$ coloring of $(G_1 - vv_1) \lor (G_2 - vv_2)$. But in any $k - 1$ coloring of $G_1 - vv_1$, $v$ and $v_1$ have the same color. Similarly, in any $k - 1$ coloring of $G_2 - vv_2$, $v$ and $v_2$ have the same color. Therefore in any $k - 1$ coloring of $(G_1 - vv_1) \lor (G_2 - vv_2)$ we see that $v_1$ and $v_2$ are the same color. This means that no $k - 1$ coloring of $(G_1 - vv_1) \lor (G_2 - vv_2)$ is the restriction of a $k - 1$ coloring of $(G_1 - vv_1) \lor (G_2 - vv_2) + v_1v_2$, therefore $(G_1 - vv_1) \lor (G_2 - vv_2) + v_1v_2$ has chromatic number at least $k$.

We can color $(G_1 - vv_1) \lor (G_2 - vv_2) + v_1v_2$ with $k$ colors as follows: color $(G_1 - vv_1)$ with $k - 1$ colors and color $(G_2 - vv_2)$ with the same $k - 1$ colors, choosing the same color for $v$ in both graphs. (This is easily possible since we can permute the colors in any coloring.) This gives a $k - 1$ coloring of $(G_1 - vv_1) \lor (G_2 - vv_2)$. Then we can change $v_1$ to a new color and add edge $v_1v_2$ to a $k$ coloring of $(G_1 - vv_1) \lor (G_2 - vv_2) + v_1v_2$.

Let us show that $(G_1 - vv_1) \lor (G_2 - vv_2) + v_1v_2$ is $k$-critical. It suffices to show that removing any edge $e$ reduces the chromatic number. Suppose $e$ is in $(G_1 - vv_1)$ or $(G_2 - vv_2)$. Without loss of generality (by the symmetry of the construction) assume $e$ is in $(G_1 - vv_1)$. Then $(G_1 - e)$ can be $k - 1$ colored. Notice that $(G_1 - e)$ has edge $vv_1$, so in this coloring $v$ and $v_1$ have different colors. Restrict this to a coloring of $(G_1 - vv_1 - e)$ and combine it with a $k - 1$ coloring of $(G_2 - vv_2)$ as above where $v$ and $v_2$ have the same color. Then edge $v_1v_2$ can be safely added giving a $k - 1$ coloring of $(G_1 - vv_1) \lor (G_2 - vv_2) + v_1v_2 - e$. The other possibility is that $e = v_1v_2$. In this case, we have already seen a $k - 1$ coloring of $(G_1 - vv_1) \lor (G_2 - vv_2)$, so we are done.

**Problem 5**

For even $n \geq 4$ we can construct a 4-critical graph with $n$ vertices as the join (defined in Problem 3) of a 1-critical graph and a 3-critical graph. For our 1-critical graph we take a single vertex with no edges, and for our 3-critical graph we take the $n - 1$ cycle.

For odd $n \geq 7$ we can construct a 4-critical graph by applying the procedure described in Problem 4 to a 4-critical graph on 4 vertices and a 4-critical graph on $n - 3$ vertices.
Suppose there is a 4-critical graph with 5 vertices. Then it must have a 4-coloring with colors \(A, B, C, \text{ and } D\). Each color must be used, so one color must be used twice. Without loss of generality, say \(A\) is used twice and each of the other colors is used only once. The vertex colored \(B\) must have an edge to a vertex of each other color, otherwise there would be a coloring with fewer colors. The same is true for the \(C\) vertex and the \(D\) vertex. At least one of the vertices colored \(A\) must connect to the \(B\) vertex, the \(C\) vertex, and the \(D\) vertex because otherwise each \(A\) vertex could be recolored to the color it wasn’t adjacent to. Therefore, we have constructed \(K_4\) as a subgraph of our 5-vertex 4-critical graph. This is a contradiction.

**Problem 6**

The chromatic polynomial for both graphs is

\[
k^5 - 10k^4 + 41k^3 - 84k^2 + 84k - 32
\]

The decomposition is shown on the final page.

**Problem 7**

We will show by induction the polygon on \(n\) vertices \(P_n\) has \(\pi_k(P_n) = (k-1)^n + (-1)^n(k-1)\).

For our base case, it is easy to check that \(k(k-1)(k-2) = (k-1)^3 - (k-1)\).

Suppose that \(\pi_k(P_n) = (k-1)^n + (-1)^n (k-1)\). Then for any edge \(e\) of \(P_{n+1}\), \(P_{n+1} - e\) is a tree \(T_{n+1}\) on \(n+1\) vertices, and \(P_{n+1} \cdot e = P_n\). Thus \(\pi_k(P_{n+1}) = \pi_k(T_{n+1}) - \pi_k(P_n) = k(k-1)^n - (k-1)^n - (-1)^n(k-1) = (k-1)^n + 1 + (-1)^{n+1}(k-1)\).

**Problem 8**

Notice that if a graph has a loop then it cannot be colored, and if a graph has more than one edge between a single pair of vertices, no additional restriction is placed on coloring of that graph (as compared to the same graph with the duplicate edge(s) removed). Therefore, the chromatic polynomial of any non-simple graph is the same as the chromatic polynomial of a corresponding simple graph. Thus, it is sufficient to consider simple graphs.

We claim that for a graph \(G\) with \(n\) vertices and \(m\) edges, \(\pi_k(G) = k^n - mk^{n-1} + O(k^{n-2})\). (Here \(O(k^{n-2})\) is used as shorthand for an unspecified polynomial of degree at most \(n-2\). Each time it is used it may mean a different polynomial.) We proceed by strong induction on \(m + n\).

For \(m = 0\), we trivially have \(\pi_k(G) = k^n\). Otherwise, if the inductive hypothesis holds for graphs with strictly fewer vertices and/or edges than \(G\), then pick any edge \(e\) in \(G\). \(\pi_k(G) = \pi_k(G - e) - \pi_k(G \cdot e) = (k^n - (m-1)k^{n-1}) - k^{n-1} + O(k^{n-2}) = k^n - mk^{n-1} + O(k^{n-2}).\)
Problem 9

$G$ is connected, so it has a spanning tree $T$. $T$ is a subgraph of $G$, so every valid coloring of $G$ inducing a unique valid coloring on $T$. Therefore $G$ cannot have more colorings than $T$:

$$\pi_k(G) \leq \pi_k(T) = k(k - 1)^{n-1}$$
First we show the two graphs have the same chromatic polynomial:

\[ \text{This is a nice decomposition that can be evaluated by (careful) inspection:} \]

\[ \text{Here is a full decomposition into polygons and trees (with isolated vertices):} \quad \text{(0's indicate the next edge to be removed)} \]

\[ \text{=} \]

\[ \left( k^2 - 6k + 9 \right) \left[ (k-1)^4 \cdot (k-1) \right] + (k-1) \left[ (k-1)^3 \cdot (k-1) \right] + (k-3) \left[ k(k-1)^2 \right] \]